

# Lecture #15 Review of Landau Damping; Cold Beam Instabilities Hawes ①

## I. Review: Landau Damping of Langmuir Waves

### A. Electrostatic Approximation: Vlasov-Poisson System

$$1. \frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla f_s - \frac{q_s}{m_s} \nabla \phi \cdot \frac{\partial \mathbf{v}}{\partial z} = 0$$

$$2. -\nabla^2 \phi = \frac{1}{\epsilon_0} \leq S d^3 v q_s f_s$$

### B. Laplace Transform Approach by Landau

#### 1. Linearization, Fourier Transform in Space, Laplace Transform in Time

$$\tilde{\phi}_1(\underline{k}, p) = \frac{N(\underline{k}, p)}{D(\underline{k}, p)} \quad \text{where } N(\underline{k}, p) = -i \frac{q_s k_b}{\epsilon_0 k^3} \int_{-\infty}^{\infty} dz \frac{f_s(z)}{V_z - ip/k}$$

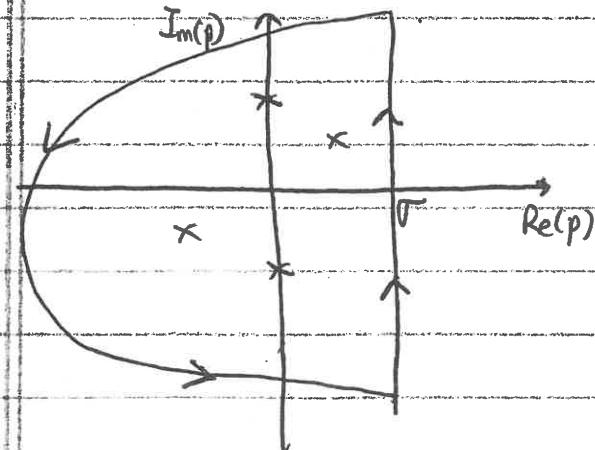
and the Dispersion Relation is  $D(\underline{k}, p) = 0$  where

$$D(\underline{k}, p) = 1 - \sum_s \frac{c_s k^2}{k^2} \int_{-\infty}^{\infty} dz \frac{\partial f_s / \partial z}{V_z - ip/k}$$

$D(\underline{k}, p) = 0$  gives  
normal modes of  
system.

#### 2. To solve, the Inverse Laplace Transform use

$$a. \phi_1(\underline{k}, t) = \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} dp \tilde{\phi}_1(\underline{k}, p) e^{pt}$$



b. We evaluate this complex contour integral by closing the contour at  $\text{Re}(p) \rightarrow -\infty$  and using the residue theorem.

c. To do so, we use analytically continue  $\tilde{\phi}_1(\underline{k}, p)$  from  $\text{Re}(p) > 0$  to  $\text{Re}(p) < 0$ .

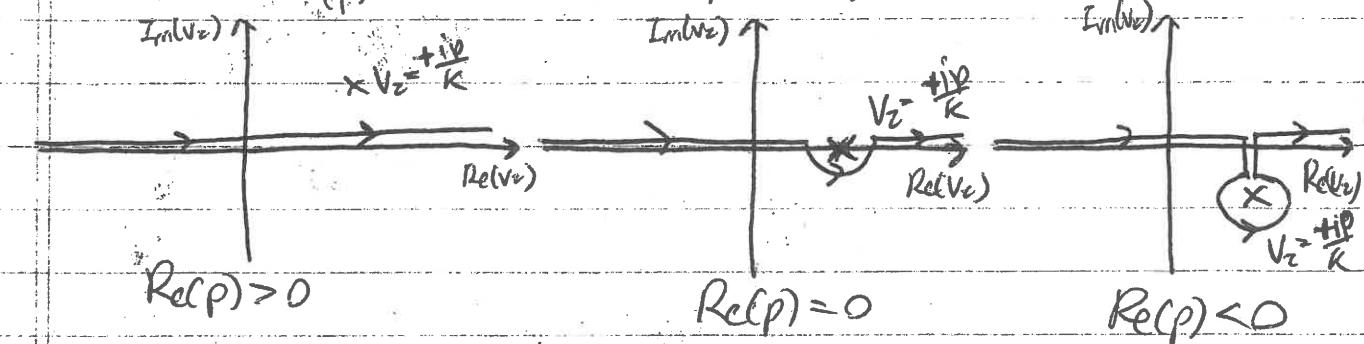
## Lecture #15 (Continued)

Haves ②

### I. A. (Continued)

3. How do we analytically continue  $D(\underline{k}, p) = 1 - \sum_s \frac{c_s p^2}{\omega_s^2} \int_{-\infty}^{\infty} dv_z \frac{\partial f_{s0}}{\partial v_z} \frac{e^{ipv_z}}{v_z - \frac{i\epsilon}{k}}$

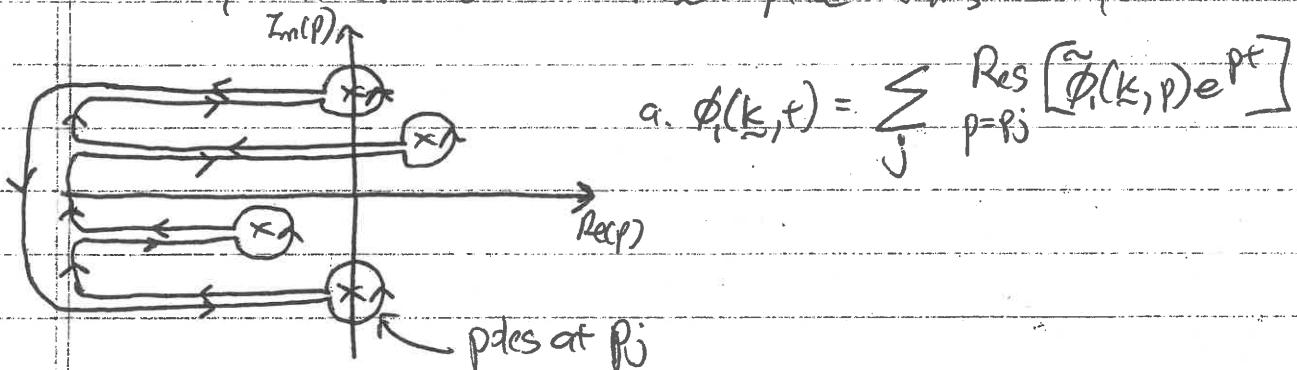
From  $\text{Re}(p) > 0$  to  $\text{Re}(p) < 0$ ?



$\Rightarrow$  Contour always passes below poles!

This definition yields an analytic form of  $\tilde{\phi}(\underline{k}, p)$  over entire complex  $p$ -plane (except for poles, of course).

4. We may then evaluate the Inverse Laplace Transform by Residue Theorem



$$a. \phi_i(\underline{k}, t) = \sum_j \text{Res}_{p=p_j} [\tilde{\phi}_i(\underline{k}, p) e^{pt}]$$

### C. Landau Damping of Waves

i. For a Cauchy Distribution of electrons with stationary ions,

$$a. F_0(\frac{p}{\omega}) = \frac{C}{\pi} \left( \frac{1}{\omega^2 + p^2} \right)$$

$$b. \text{Dispersion Relation: } D(\underline{k}, p) = 1 + \frac{C \omega^2}{(p + ikC)^2} = 0$$

$$c. \text{Wave Solutions with } \omega = \pm \omega_p \text{ and } \gamma = -ikC$$

↑ Landau damping due to  $\text{Re}(\gamma) < 0$  (or  $\text{Im}(\omega) < 0$ )

## I.C. (Continued)

## 2. Maxwellian Distribution:

## a. The Plasma Dispersion Function

$$Z(\xi_s) = \int_C^{\infty} \frac{dz}{\pi^{1/2}} \frac{e^{-\xi_s^2}}{z - \xi_s}$$

where  $\xi_s = \frac{ip}{kv_{es}}$  or  $\xi_s = \frac{\omega}{kv_{es}} + i \frac{\delta}{kv_{es}}$

b. Using this function, the Langmuir Wave Dispersion Relation can be written,

$$D(k, p) = 1 + \frac{1}{k^2 \lambda_{de}^2} [1 + \xi_e Z(\xi_e)] = 0$$

c. In the high phase velocity limit,  $|\xi_e| \gg 1$ , or  $\frac{\omega}{k} \gg v_{te}$ , we find

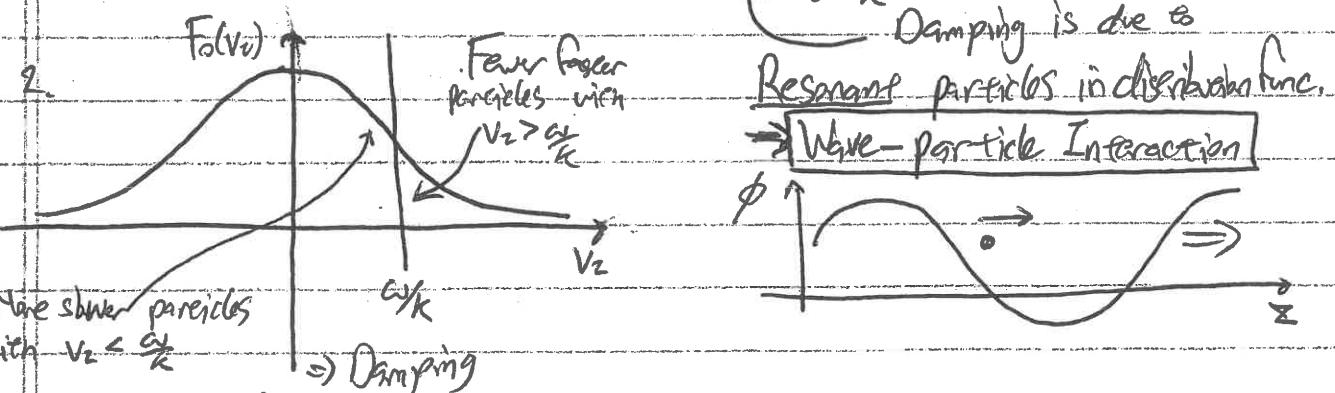
$$\omega^2 = \omega_{pe}^2 + \frac{3}{2} k^2 v_{te}^2 \quad \text{and} \quad \gamma = -\frac{1}{8} \frac{\omega_{pe}}{|k|^3 \lambda_{de}^3} e^{\left(-\frac{1}{2k^2 \lambda_{de}^2} - \frac{3}{2}\right)}$$

↑ Landau Damping.

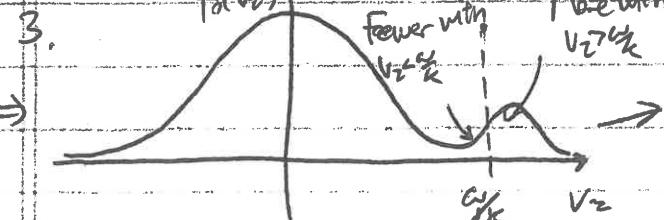
## D. Physical Interpretation of Landau Damping

1. In the Weak Growth Rate Approximation,  $|\gamma| \ll |\omega|$ ,

$$\gamma = \pi \frac{k}{|k|} \frac{1}{\partial F_0 / \partial v} \lesssim \frac{\omega_{pe}^2}{k^2} \left. \frac{\partial F_0}{\partial v} \right|_{v=0}$$



Due to beam of base particles.



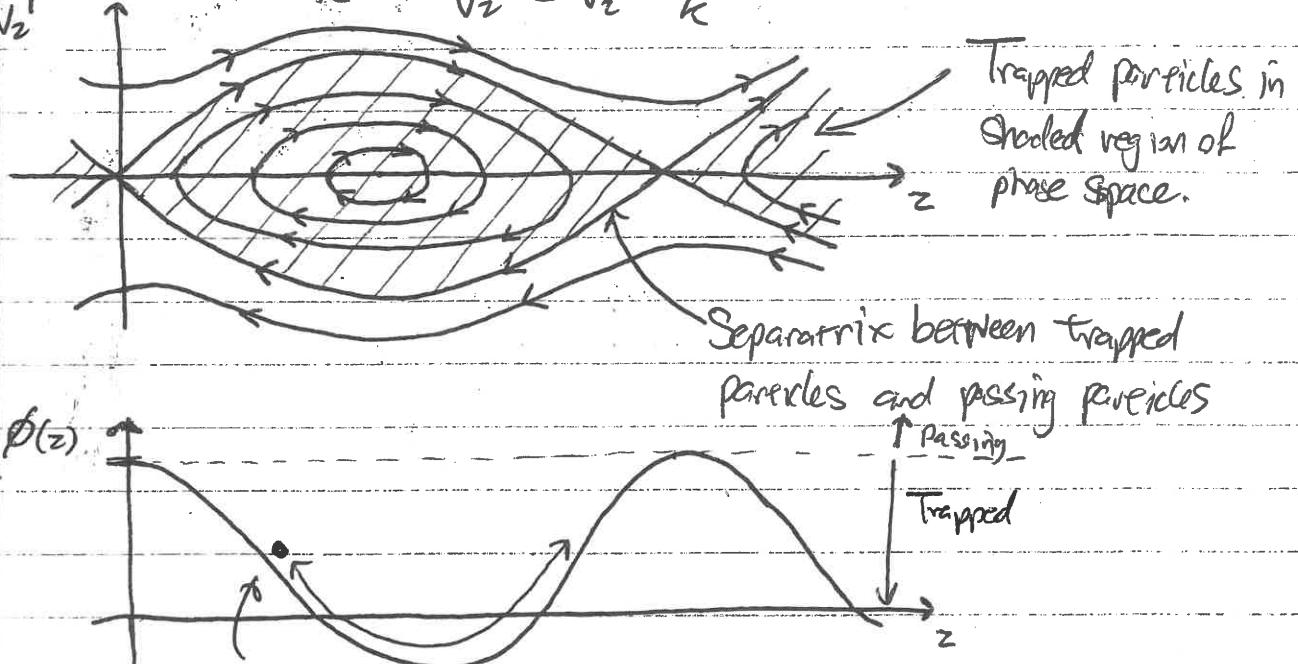
Bump-on-tail Instability:  
Leads to unstable growth of E fluctuations from free energy in  $F_0(v_z)$ .

## II. Phase Mixing Interpretation

### A. Phase Space Pde ( $v_z, z$ )

1. Transform to frame of reference moving at phase velocity of

$$\text{the wave } \frac{\omega}{k} : v_z' = v_z - \frac{\omega}{k}$$

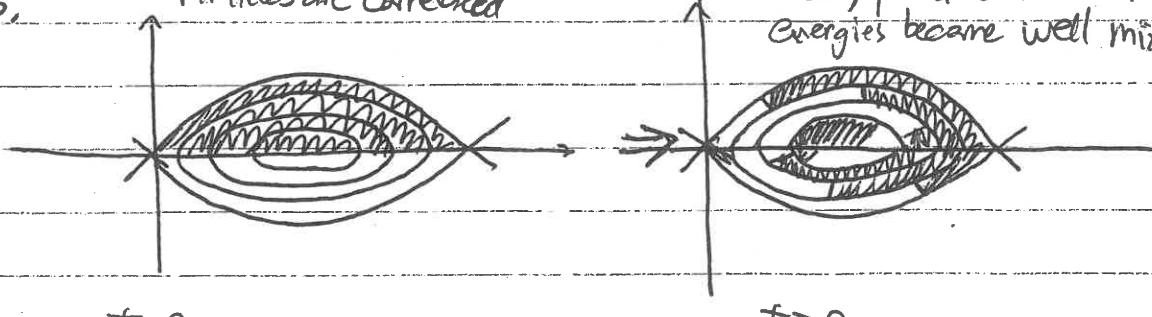


Particles trapped in potential well (mainly at phase speed  $\frac{\omega}{k}$ )

2. Bounce Frequency of trapped particles decreases with increasing energy, reaching zero at Separatrix.

3. Initially  
Particles are correlated

Later, particles at different energies became well mixed.



a. Since potential  $-\nabla^2\phi = \frac{1}{2}\rho d^3v q_s B_s$  depends on integration over  $B_s$ , this phase mixing leads to an averaging over the integral  $\Rightarrow$  damping of the waves

### III. Cold Beam Instabilities

#### A. Cold Beam Distribution:

1. A beam with number density  $n_j$  and velocity  $\mathbf{v} = v_j \hat{z}$  is given by  $F_0(\mathbf{v}) = n_j \delta(v_x) \delta(v_y) \delta(v_z - v_j)$

2. Since all particles have the same velocity, the beam has zero temperature  $\Rightarrow$  no thermal spread of velocities

3. For a species  $s$ , we can have multiple beams,

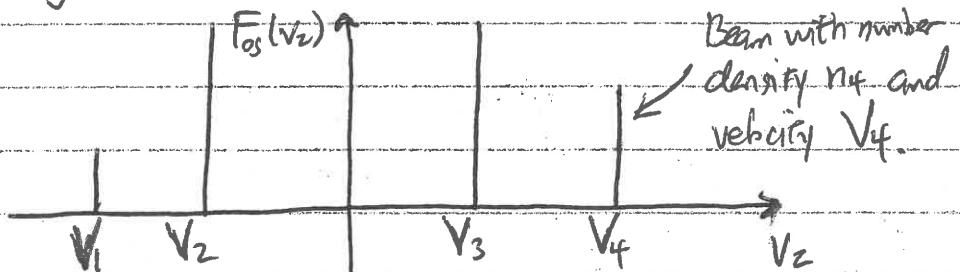
$$F_{os}(\mathbf{v}) = \sum_j n_j \delta(v_x) \delta(v_y) \delta(v_z - v_{js})$$

4. The one-dimensional distribution function is written

$$F_{os}(v_z) = \frac{1}{n_{os}} \int_{-\infty}^{\infty} F_{os}(\mathbf{v}) dv_x dv_y = \sum_j \frac{n_j}{n_{os}} \delta(v_z - v_{js})$$

where  $n_o = \sum_j n_j$  is the total number density of all beams.

#### B. Distribution:



#### B. Dispersion Relation for Electrostatic Waves

1. The Laplace-Fourier Solution for electrostatic waves gives a dispersion relation

$$D(k, p) = 1 - \sum_s \frac{\omega_{ps}^2}{k^2} \int_{-\infty}^{\infty} dv_z \frac{F_{os}}{(v_z - ip/k)^2} = 0$$

2. Using  $\sum_j \omega_{pj}^2 \frac{n_j}{n_{os}} = \sum_j \frac{n_{os} q_s^2}{60 \text{ ms}} \frac{n_j}{n_{os}} = \sum_j \omega_{pj}^2$  where  $\omega_{pj}^2 = \frac{n_j q_s^2}{60 \text{ ms}}$ ,

we obtain:  $D(k, p) = 1 - \sum_j \frac{\omega_{pj}^2}{k^2} \left( \frac{1}{(v_{js} - ip/k)^2} \right) = 0$

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III.B (Continued)

3. NOTE: Here we take a complex  $\omega = i\gamma$

(we may write  $\omega = \omega_r + i\gamma$  to denote real & imaginary parts).

$$\text{So } D(k, \omega) = 1 - \sum_j \frac{\omega_p j^2}{k^2} \left( V_j - \frac{\omega}{k} \right)^2 = 0$$

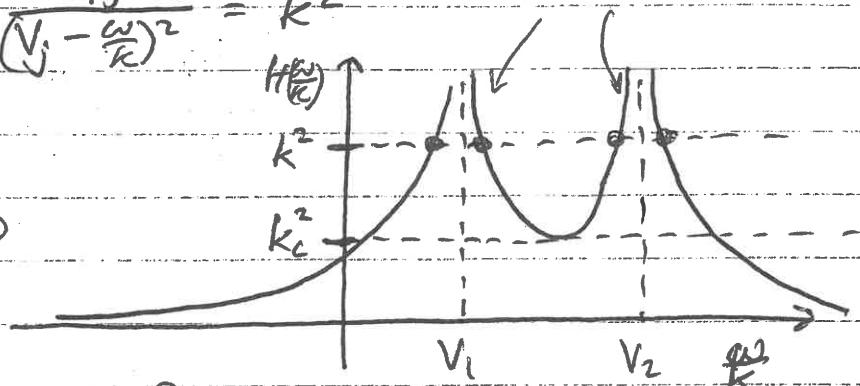
4. Solutions occur when

$$H\left(\frac{\omega}{k}\right) = \sum_j \frac{\omega_p j^2}{\left(V_j - \frac{\omega}{k}\right)^2} = k^2$$

Two solutions per beam.

a. We may plot this as

$$\text{For two beams} \Rightarrow$$



b. For  $k > k_c$ , there exist four real solutions  $\Rightarrow \gamma \neq 0$ .

c. But, for  $k < k_c$ , one obtains two real & two imaginary solutions.

Since  $H\left(\frac{\omega}{k}\right)$  is real, the roots come in complex conjugate pairs

$$\begin{aligned} \omega_+ &= \omega_r + i\gamma \\ \omega_- &= \omega_r - i\gamma \end{aligned} \quad ] \text{Here we assume } \gamma > 0.$$

d. Thus, the root  $\omega_+ = \omega_r + i\gamma$  gives a time dependence

$$\sim e^{-i\omega_r t} e^{i\gamma t}$$

Growth of this aux solution.  $\Rightarrow$  UNSTABLE

5.a. Single Beam leads to  $\omega = \pm \omega_p - kV \Rightarrow$  Always stable.

b. When two (or more) beams are present, for sufficiently long wave lengths  $k < k_c$ , the plasma is always unstable.

Two Stream Instability

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### III. Continued

C. Two Stream Instability: Equal and opposite beams.

1. Take  $n_1 = n_2 = n_0$        $V_1 = -V_2 = V$ .

2. Thus

$$D(k, \omega) = 1 - \frac{\omega_p^2}{(\omega - kV)^2} - \frac{\omega_p^2}{(\omega + kV)^2} = 0$$

$\omega_p^2 = n_0 q c^2 / 6 m s$

3. a. This equation is quadratic in  $\omega^2$  and can be expressed as

$$\omega^4 - \omega^2 [2(\omega_p^2 + k^2 V^2)] - k^2 V^2 (2\omega_p^2 - k^2 V^2) = 0$$

b. Solution:

$$\omega^2 = \omega_p^2 + k^2 V^2 \pm \sqrt{\omega_p^4 + 4\omega_p^2 k^2 V^2}$$

positive definite

positive definite

for solution with negative sign, when

$$\omega_p^2 + k^2 V^2 < (\omega_p^4 + 4\omega_p^2 k^2 V^2)^{\frac{1}{2}},$$

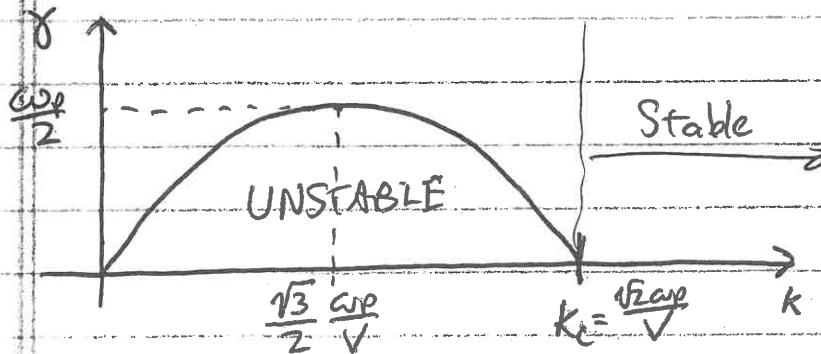
then  $\omega^2 < 0$  and the solution  $\omega = \pm i\gamma$ , leading to unstable growth.

b. Condition for this is

$$k < k_c = \sqrt{2} \frac{\omega_p}{V}$$

c. The growth rate is

$$\gamma = \left[ (\omega_p^4 + 4k^2 V^2 \omega_p^2)^{\frac{1}{2}} - (\omega_p^2 + k^2 V^2) \right]^{\frac{1}{2}}$$



## Lecture #15 (Continued)

Homework 8

### III. D. Weak Beam Approximation

1. Take a low density electron beam in a plasma at rest,  $n_b \ll n_0$ .

2. For  $\omega_p^2 = \frac{n_0 e^2}{\epsilon_0 m}$ , we get  $D(k, \omega) = 1 - \frac{\omega_p^2}{\omega^2} - \frac{\epsilon \omega_b^2}{(kV - \omega)^2} = 0$

3. Ordering:  $\epsilon = \frac{\omega_p^2}{\omega_p^2} \ll 1$ : Let  $\omega = \omega_0 + \epsilon \omega_1$ .

$$a. \left[ 1 - \frac{\omega_p^2}{(\omega_0 + \epsilon \omega_1)^2} \right] (kV - (\omega_0 + \epsilon \omega_1))^2 = \epsilon \omega_b^2$$

$$b. \text{Expanding: } \frac{\omega_p^2}{\omega_0^2 (1 + \frac{\epsilon \omega_1}{\omega_0})^2} \approx \frac{\omega_p^2}{\omega_0^2} \left( 1 - \frac{2\epsilon \omega_1}{\omega_0} \right)$$

$$4. G(1): \left( 1 - \frac{\omega_p^2}{\omega_0^2} \right) (kV - \omega_0)^2 = 0 \Rightarrow \omega_0 = \pm \omega_p \quad k = \frac{\pm \omega_p}{V}$$

5. To solve for  $G(\epsilon)$ , substitute solution  $\omega_0$  and  $k$ ,

$$a. \left[ 1 - \frac{\omega_p^2}{\omega_p^2} \left( 1 - \frac{2\epsilon \omega_1}{\omega_0} \right) \right] [kV - \omega_p - \omega_1]^2 = \omega_b^2$$

$$\Rightarrow \boxed{\omega_1^3 = \frac{\omega_b^2 \omega_p}{2}}$$

$$b. \text{Three roots are: } \omega_1 = \left( \frac{1}{2} \right)^{\frac{1}{3}} \omega_b^{\frac{2}{3}} \omega_p^{\frac{1}{3}} \text{ and } \omega_1 = \left( \frac{1}{2} \right)^{\frac{1}{3}} \omega_b^{\frac{2}{3}} \omega_p^{\frac{1}{3}} \left( -\frac{1}{2} \pm i\frac{\sqrt{3}}{2} \right)$$

6. Root with positive sign gives unstable growth with

$$a. \omega_r = \omega_p \left[ 1 - \frac{1}{2} \left( \frac{1}{2} \right)^{\frac{1}{3}} \left( \frac{n_b}{n_0} \right)^{\frac{1}{3}} \right] \text{ and } \gamma = \frac{\sqrt{3}}{2} \left( \frac{1}{2} \right)^{\frac{1}{3}} \left( \frac{n_b}{n_0} \right)^{\frac{1}{3}} \omega_p$$

$$b. \text{Growth rate } \gamma \propto n_b^{\frac{1}{3}}$$

### E. Buneman Instability

a. Background of cold ions at rest

b. Cold "Beam" of electrons moving through ions,  $n_e = n_i = n_0$

2. Dispersion Relation:  $D(k, \omega) = 1 - \frac{\omega_{pi}^2}{\omega^2} - \frac{\omega_{pe}^2}{(kV - \omega)^2} = 0$

$$3. \text{Yields } \omega_r = \frac{1}{2} \left( \frac{m_e}{2m_i} \right)^{\frac{1}{3}} \omega_{pe} \quad \gamma = \frac{\sqrt{3}}{2} \left( \frac{m_e}{2m_i} \right)^{\frac{1}{3}} \omega_{pe}$$

This fast growing instability occurs when cold electrons stream through cold ions.