

Whistler Waves

Adam

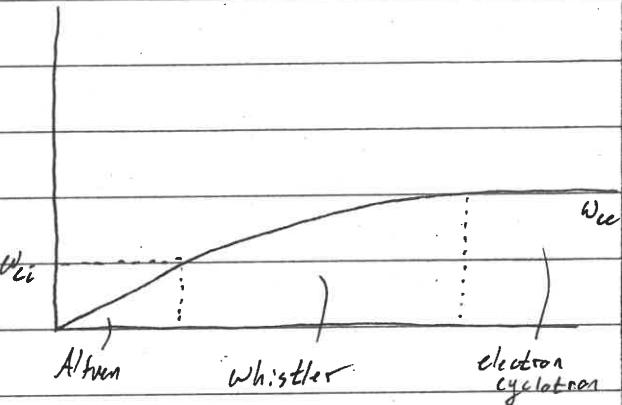
Limits (parallel propagation: $\mathbf{k} \parallel \mathbf{B}$)

- Whistler waves are right-handed polarized waves ($n^2 = R$)
 - subset of electron cyclotron waves, with limits on ω

$$\omega_{ci} \ll \omega \ll |\omega_{ce}| \ll \omega_{pe}$$

$$E_i = (E_0, iE_\phi, 0)$$

$$\omega = \frac{n^2 c^2}{\omega_{pe}^2} |\omega_{ce}|$$



Ion Current

using eq of motion to find magnetization from Lorentz force

$$\frac{q_i B_0}{-i \omega M_i} \cdot V_i \times B = \omega_{ci}, \text{ the ratio of } \frac{\omega_{ci}}{\omega} \text{ determines how negligible } V_i \times B \text{ is}$$

$$\Rightarrow \frac{\omega_{ci}}{n^2 c^2 / \omega_{ce}} \cdot \omega_{pe} = \frac{q_i B_0}{n^2 c^2 M_i} \cdot \frac{m_e}{q_i B_0} \cdot \omega_{pe}^2 \Rightarrow m_e \ll 1 \therefore \text{not magnetized}$$

$$\text{Current: } j_i = n_i q_i V_i = n_i q_i \cdot \frac{q_i}{-i \omega M_i} \cdot E_i = \frac{q_i^2 n_i q_i}{E_i M_i} \cdot \frac{\omega_{pe}^2}{n^2 c^2 / \omega_{ce}}$$

$$\Rightarrow \frac{i E_0 \omega_{pi}^2 \cdot \omega_{pe}^2}{n^2 c^2 / \omega_{ce}} = \boxed{\frac{i E_0 \omega_{pi}^2}{\omega}}$$

remember, $\omega_{pe} \ll \omega$
so ion current is
very small

Electron Current

- ° We can use the same procedure to check for magnetization:

$$\Rightarrow \frac{i w_{ce}}{\omega} \rightarrow \frac{i w_{ce} \omega_p^2}{\kappa c^2 \omega_c} \Rightarrow \omega_p^2 \gg 1 \quad \therefore \text{electrons are magnetized}$$

- ° We can now use the same procedure in the notes, using drift to find current

$$\Rightarrow j_e = n_e q_e v_e = \underbrace{\sim \sim}_{\text{drift}} + \frac{E_i \times \hat{B}}{B_0} - \frac{i w_e E_i}{w_{ce} B_0} \quad \text{we also multiply by } \frac{q_e}{2e}, \frac{m_e}{m_i}, \frac{E_0}{E_i}$$

This term is dependant on E_z , but $E_z = 0$ for Whistler waves

$$\Rightarrow j_e = \frac{q_e m_e E_0}{q_e m_i E_0} \frac{q_e n_e}{B_0} (E_i \times \hat{B}) - \frac{q_e^2 m_e \omega}{q_e m_i E_0} \frac{i w_e E_i}{w_{ce} B_0} = \frac{E_0 \omega_p^2}{w_{ce}} F_{x, B} - \frac{i E_0 \omega_w \omega_p^2}{w_{ce}^2} F_z$$

$$\Rightarrow j_e = \frac{E_0 \omega_p^2}{w_{ce}} \tilde{F}_{x, B} - \frac{i E_0 \omega_p^2}{\kappa c^2 / w_{ce}} \frac{\omega_p^2}{w_{ce}} F_z$$

stays constant

per plasma characteristics

Varies w/respect to κ \Rightarrow as $\kappa \rightarrow \infty$

the current becomes more similar to electron cyclotron

as $\kappa \rightarrow 0$, current approaches Alfvén behavior

Limiting Behavior

- Looking at Faraday's Law, and our characteristics on transverse:

$$\cancel{kx} \cancel{E} \neq 0, \therefore E_i = E_0(\hat{x} + i\hat{y}), B_i = -\frac{i k E_0}{\omega}(\hat{x} + i\hat{y}), \boxed{\text{Electromagnetic}}$$



There is a relationship between ω and only \vec{B}_i , not \vec{E} .

- We can find the frequency mode by using the relationship of $n^2 = R$:

$$\Rightarrow \frac{k^2 c^2}{\omega^2} = \frac{(w + w_i)(w - w_r)}{(w + w_{ci})(w + w_{ci})}, \quad \text{where we can apply the following limits}$$

$$w_{ci} \ll w \ll w_{ce} < w_{pe}$$

$$\text{and: } w_r w_i = w_{pe}^2 + w_{ce} w_{ci},$$

$$\Rightarrow \frac{w^2 - w w_r + w w_i - w_r w_i}{w_{ce} w}$$

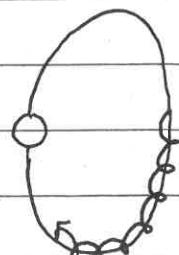
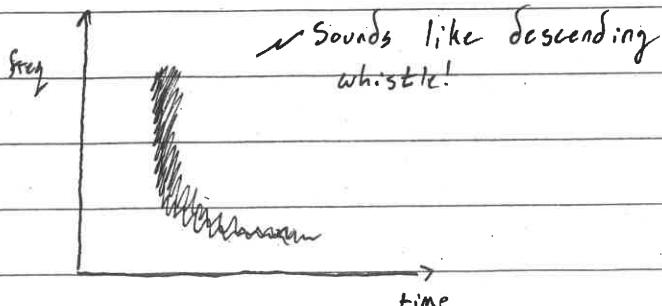
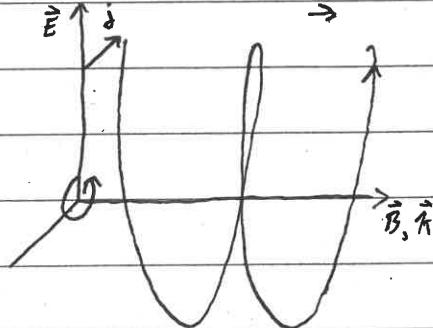
$$-w_r + w_i \approx -|w_{ce}|$$

$$\Rightarrow (w^3 + w^2(w_i - w_r) - w w_i w_r) \cdot \frac{1}{w_{ce}} \Rightarrow \frac{w^2}{w_{ce}} + w^2 - \frac{w \cdot w_{pe}^2}{w_{ce}} - \frac{w w_i w_{ci}}{w_{ce}}$$

$$\Rightarrow k^2 c^2 = -w^2 - \frac{w \cdot w_{pe}^2}{-|w_{ce}|} - w w_{ci} \Rightarrow \boxed{\frac{k^2 c^2}{w_{pe}^2} \cdot |w_{ce}| = \omega}$$

Physical Behaviors

- ions have little to no current
- electrons have net current along magnetic field
 - Also have circular orbit as \vec{E} rotates
- Wave has group velocity $v_g = \frac{sw}{jk}$
 - Whistlers can be generated by lightning (many freq)
 - High freq travel faster, arrive first when measured



Travel along Earth's \vec{B}

Cold Plasma Alfvén Wave

Jan 31st, 2023

I. Basic Settings

1. 2-fluid theory
2. Cold: $T_s = 0$
3. ion-electron plasma
4. Uniform magnetized
5. Quasi-neutrality: $\sum_s n_{s0} q_{s0} = 0 \rightarrow \sum_s \omega_{ps}^2 / \omega_{cs}^2 = 0$ and $\omega_{pi}^2 \omega_{ce} + \omega_{pe}^2 \omega_{ci} = 0$

II. Linear Dispersion Relation

1. Linearization

$$n_s = n_{s0} + \epsilon n_{s1}, \vec{U}_s = \epsilon \vec{U}_{s1}, \vec{E} = \epsilon \vec{E}_1, \vec{B} = \vec{B}_0 + \epsilon \vec{B}_{s1}, \rho_q = \epsilon \rho_{q1}, \vec{J} = \epsilon \vec{J}_1 \quad (2.1)$$

and Fourier transform: according to

$$f(\vec{x}, t) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \tilde{f}(\vec{k}, \omega) e^{i(\vec{k} \cdot \vec{x} - \omega t)} d^3 \vec{k} d\omega, \quad (2.2)$$

We have:

$$\nabla \rightarrow i\vec{k}, \quad \frac{\partial}{\partial t} \rightarrow -i\omega \quad (2.3)$$

Linearized and Fourier transformed equations:

$$\text{Continuity Equation : } \frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \vec{U}_s) = 0 \implies \omega \tilde{n}_{s1} = n_{s0} \vec{k} \cdot \tilde{\vec{U}}_{s1} \quad (2.4)$$

$$\text{Momentum Equation : } m_s n_s \left(\frac{\partial \vec{U}_s}{\partial t} + \vec{U}_s \cdot \nabla \vec{U}_s \right) = q_s n_s (\vec{E} + \vec{U}_s \times \vec{B}) \implies -i\omega \tilde{\vec{U}}_{s1} = \frac{q_s}{m_s} (\tilde{\vec{E}}_1 + \tilde{\vec{U}}_{s1} \times \tilde{\vec{B}}_0) \quad (2.5)$$

$$\text{Ampere's Law : } \nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \implies i\vec{k} \times \tilde{\vec{B}}_1 = \mu_0 \tilde{\vec{J}}_1 - i\mu_0 \epsilon_0 \omega \tilde{\vec{E}}_1 \quad (2.6)$$

$$\text{Faraday's Law : } \nabla \cdot \vec{E} = -\frac{\partial \vec{B}}{\partial t} \implies \vec{k} \cdot \tilde{\vec{E}}_1 = \omega \tilde{\vec{B}}_1 \quad (2.7)$$

$$\text{Gauss's Law : } \nabla \cdot \vec{E} = \frac{\rho_q}{\epsilon_0} \implies i\vec{k} \cdot \tilde{\vec{E}}_1 = \frac{\rho_{q1}}{\epsilon_0} \quad (2.8)$$

$$\text{No Magnetic Monopole : } \nabla \cdot \vec{B} = 0 \implies \vec{k} \cdot \tilde{\vec{B}}_1 = 0 \quad (2.9)$$

$$\text{Charge Density : } \rho_q = \sum_s n_s q_s \implies \tilde{\rho}_{q1} = \sum_s \tilde{n}_{s1} q_s \quad (2.10)$$

$$\text{Current Density : } \vec{J} = \sum_s n_s q_s \vec{U}_s \implies \tilde{\vec{J}}_{s1} = \sum_s n_{s0} q_s \tilde{\vec{U}}_{s1} \quad (2.11)$$

$$2. \text{ From momentum equation find } \tilde{\vec{J}}_1 = \overset{\leftrightarrow}{\sigma} \cdot \tilde{\vec{E}}_1$$

$$\tilde{\vec{J}}_1 = \sum_s n_s q_s \tilde{\vec{U}}_1 = \sum_s n_s q_s \frac{q_s}{m_s} \begin{bmatrix} \frac{i\omega}{\omega^2 - \omega_{cs}^2} & \frac{-\omega_{cs}}{\omega^2 - \omega_{cs}^2} & 0 \\ \frac{\omega_{cs}}{\omega^2 - \omega_{cs}^2} & \frac{i\omega}{\omega^2 - \omega_{cs}^2} & 0 \\ 0 & 0 & \frac{i}{\omega} \end{bmatrix} \begin{bmatrix} \tilde{E}_{1x} \\ \tilde{E}_{1y} \\ \tilde{E}_{1z} \end{bmatrix} \quad (2.12)$$

$$3. \text{ From } \vec{n} \times (\vec{n} \times \vec{E}) + \overset{\leftrightarrow}{\epsilon} \cdot \vec{E} = 0 \text{ find linear dispersion relation:}$$

$$\begin{bmatrix} S - n^2 \cos^2 \theta & -iD & n^2 \sin \theta \cos \theta \\ iD & S - n^2 & 0 \\ n^2 \sin \theta \cos \theta & 0 & P - n^2 \sin^2 \theta \end{bmatrix} \begin{bmatrix} \tilde{E}_{1x} \\ \tilde{E}_{1y} \\ \tilde{E}_{1z} \end{bmatrix} = 0 \quad (2.13)$$

III. Limits of the Cold Plasma Alfvén Wave

1. $\vec{k} \parallel \vec{B}_0$, choose $\vec{k} = k\hat{z}$ and $\vec{B}_0 = B_0\hat{z}$, and $\theta = 0$

2. low frequency $\omega \ll \omega_{ci} \ll |\omega_{ce}| \ll \omega_{pe} \Rightarrow$

$$R = 1 - \sum_s \frac{\omega_{ps}^2}{\omega(\omega + \omega_{cs})} = 1 - \frac{\omega_{pi}^2 + \omega_{pe}^2}{(\omega + \omega_{ci})(\omega + \omega_{ce})} \approx 1 + \frac{\omega_{pe}^2}{\omega_{ci}|\omega_{ce}|} = 1 + \frac{\omega_{pi}^2}{\omega_{ci}^2} \approx L$$

$$S = \frac{1}{2}(R + L) \approx 1 + \frac{\omega_{pi}^2}{\omega_{ci}^2}, \quad D = \frac{1}{2}(R - L) \approx 0, \quad P \approx -\frac{\omega_{pe}^2}{\omega^2} \quad (3.1)$$

3. $\vec{k} \rightarrow 0$

IV. Mode Frequency

With the approximations above, the linear dispersion relation now is:

$$\begin{bmatrix} S - n^2 & 0 & 0 \\ 0 & S - n^2 & 0 \\ 0 & 0 & P \end{bmatrix} \begin{bmatrix} \tilde{E}_{1x} \\ \tilde{E}_{1y} \\ \tilde{E}_{1z} \end{bmatrix} = \begin{bmatrix} 1 + \frac{\omega_{pi}^2}{\omega_{ci}^2} - n^2 & 0 & 0 \\ 0 & 1 + \frac{\omega_{pi}^2}{\omega_{ci}^2} - n^2 & 0 \\ 0 & 0 & -\frac{\omega_{pe}^2}{\omega^2} \end{bmatrix} \begin{bmatrix} \tilde{E}_{x1} \\ \tilde{E}_{y1} \\ \tilde{E}_{z1} \end{bmatrix} = 0 \quad (4.1)$$

$$\Rightarrow (1 + \frac{\omega_{pi}^2}{\omega_{ci}^2} - n^2)^2 (-\frac{\omega_{pe}^2}{\omega^2}) = 0 \quad (4.2)$$

$$\Rightarrow \omega^2 = \frac{k^2 v_A^2}{1 + \frac{v_A^2}{c^2}} \approx k^2 v_A^2 \quad (4.3)$$

$$\text{with } v_A = \frac{B_0}{\sqrt{\mu_0 n_{10} m_i}} = \frac{B_0}{\sqrt{\mu_0 \rho_{10}}} \quad (4.4)$$

Hence, phase velocity $\tilde{v}_p = \omega/\tilde{k} = \pm v_A \hat{z}$, group velocity $v_g = \partial\omega/\partial\tilde{k} = \pm v_A \hat{z}$. And this mode is non-dispersive.

We choose $\tilde{E}_1 = (\tilde{E}_1, 0, 0)$ as our eigenfunction to do the following calculations.

V. Ion and Electron Current

1. Ion Current:

From eq.(2-12):

$$\begin{aligned} \tilde{J}_{i1} &= n_{i0} q_i \tilde{U}_{i1} = n_{i0} q_i \frac{q_i}{m_i} \left(\frac{i\omega \tilde{E}_1}{\omega^2 - \omega_{ci}^2} \hat{x} + \frac{\omega_{ci} \tilde{E}_1}{\omega^2 - \omega_{ci}^2} \hat{y} \right) \\ &= n_{i0} q_i \frac{q_i}{m_i} \left(\frac{1}{\omega} \frac{iE_0}{1 - (\omega_{ci}/\omega)^2} \hat{x} + \frac{1}{\omega_{ci}} \frac{E_1}{(\omega/\omega_{ci})^2 - 1} \hat{y} \right) \\ &\approx -n_{i0} q_i \frac{q_i}{m_i} \left(\frac{i\omega \tilde{E}_1}{\omega_{ci}^2} \hat{x} + \frac{\tilde{E}_1}{\omega_{ci}} \hat{y} \right) = -n_{i0} q_i \frac{i\omega \tilde{E}_1}{\omega_{ci} B_0} \hat{x} - n_{i0} q_i \frac{\tilde{E}_1}{B_0} \hat{y} \end{aligned} \quad (5.1)$$

Recall the polarization drift and ExB drift along with their Fourier transformed formula:

$$\tilde{v}_{sp} = \frac{m_s}{q_s B_0^2} \frac{d\tilde{E}}{dt} \Rightarrow \tilde{v}_{sp} = -\frac{i\omega}{\omega_{ce} B_0} \tilde{E} \quad (5.2)$$

$$\tilde{v}_E = \frac{\tilde{E}_1 \times \vec{B}_0}{B_0^2} \Rightarrow \tilde{v}_E = \frac{\tilde{E}_1 \times \vec{B}_0}{B_0^2} \quad (5.3)$$

We can see that the first term is caused by ion polarization drift. In real space, the ion polarization current is pointing towards the direction where \tilde{E}_1 is increasing. The second term is due to the ion ExB drift, pointing to the direction of $-\hat{y}$.

Compare the magnitude of these two terms:

$$\frac{|n_{i0} q_i \frac{i\omega \tilde{E}_1}{\omega_{ci} B_0}|}{|n_{i0} q_i \frac{\tilde{E}_1}{B_0}|} \sim \left| \frac{\omega}{\omega_{ci}} \right| \ll 1 \quad (5.4)$$

, meaning the ExB drift current dominates.

2. Electron Current: Similarly,

$$\tilde{\tilde{J}}_{e1} = n_{e0}q_e \tilde{\tilde{U}}_{e1} = -n_{e0}|q_e| \frac{i\omega \tilde{E}_1}{|\omega_{ce}|B_0} \hat{x} + n_{e0}|q_e| \frac{\tilde{E}_1}{B_0} \hat{y} \quad (5.5)$$

The first term is electron polarization drift current, pointing towards the same direction as the ion polarization drift. Also note that if we only focus on the electron velocity, we will find that electrons are moving to the opposite direction of the electron polarization current. The second term is the electron ExB drift current, flowing to $+\hat{y}$ direction, with electrons moving along $-\hat{y}$.

Compare the electron velocity and ion velocity, we will find that they are moving together.

3. Total Current:

$$\begin{aligned} \tilde{\tilde{J}}_1 &= \tilde{\tilde{J}}_{i1} + \tilde{\tilde{J}}_{e1} \\ &= -n_{i0}q_i \frac{i\omega \tilde{E}_1}{\omega_{ci}B_0} \hat{x} - \cancel{n_{i0}q_i \frac{\tilde{E}_1}{B_0} \hat{y}} - n_{e0}|q_e| \frac{i\omega \tilde{E}_1}{|\omega_{ce}|B_0} \hat{x} + \cancel{n_{e0}|q_e| \frac{\tilde{E}_1}{B_0} \hat{y}} \\ &= -n_{i0}q_i \frac{i\omega \tilde{E}_1}{B_0} \left(\frac{1}{\omega_{ci}} + \frac{1}{|\omega_{ce}|} \right) \hat{x} \end{aligned} \quad (5.6)$$

Ion and electron ExB drift currents are canceled by each other. Only polarization drifts are left to contribute to the total current, which points to the $+\hat{x}$ direction in real space. Note that $-i$ shows up in the Fourier coefficient $\tilde{\tilde{J}}_1$ means in the position space, the current $\tilde{\tilde{J}}_1$ is $\pi/2$ out of phase compared to the perturbed electric field \tilde{E}_1 .

VI. Limiting Behavior

Recall that the mode frequency $\omega = \pm k v_A$, this means the order of infinitesimal of ω and k are the same as long as v_A is constant, and $\vec{k} \rightarrow 0$ is a natural consequence of low frequency.

Make use of other equations to study its behavior:

- From Gauss's Law eq.(2.8) $\vec{k} \cdot \tilde{\tilde{E}}_1 = 0$, meaning wave is transverse.
- From Faraday's Law eq.(2.7) solve for $\tilde{\tilde{B}}_1 = (\tilde{B}_{1x}, \tilde{B}_{1y}, \tilde{B}_{1z})$:

$$\begin{aligned} \vec{k} \times \tilde{\tilde{E}}_1 &= \omega \tilde{\tilde{B}}_1 \\ \implies \tilde{\tilde{B}}_1 &= \frac{k \tilde{E}_1}{\omega} \hat{z} \times \hat{x} = \pm \frac{\tilde{E}_1}{v_A} \hat{y} = \tilde{B}_1 \hat{y} \end{aligned} \quad (6.1)$$

Hence, the perturbed magnetic field is: $\tilde{\tilde{B}}_1 = (0, \pm \tilde{E}_1/v_A, 0)$.

- Examine terms in the Ampere's Law eq.(2.6):

$$\begin{aligned} \vec{k} \times \tilde{\tilde{B}}_1 &= -i\mu_0 \tilde{\tilde{J}}_1 - \frac{\omega}{c^2} \tilde{\tilde{E}}_1 \\ &= -i\mu_0 \left(-n_{i0}q_i \frac{i\omega \tilde{E}_1}{B_0} \left(\frac{1}{\omega_{ci}} + \frac{1}{|\omega_{ce}|} \right) \right) \hat{x} - \frac{\omega}{c^2} \tilde{E}_1 \hat{x} \\ &\approx -\frac{\omega \omega_{pi}^2 \tilde{E}_1}{c^2 \omega_{ci}^2} \hat{x} - \frac{\omega}{c^2} \tilde{E}_1 \hat{x} \end{aligned} \quad (6.2)$$

The magnitude of two terms:

$$\frac{\left| \frac{\omega \omega_{pi}^2 \tilde{E}_1}{c^2 \omega_{ci}^2} \right|}{\left| \frac{\omega}{c^2} \tilde{E}_1 \right|} = \frac{\omega_{pi}^2}{\omega_{ci}^2} \gg 1 \quad (6.3)$$

Hence, we can safely neglect the displacement term.

In conclusion, the electromagnetic fluctuations propagating along the unperturbed magnetic field direction under the limit of $\vec{k} \rightarrow 0$, to the lowest order, is acting like a light wave.

VII. Physical Description

EM Wave:

- Transverse motion: perturbations \vec{E}_1 and \vec{B}_1 are perpendicular to the wave propagation direction.
- The Poynting flux vector $\vec{S} = (1/\mu_0)\vec{E} \times \vec{B}$ points the direction of flowing energy, which is $+\hat{z}$ in this case.
- The velocity of the magnetic field line is $\pm v_A B_{1y}/B_0$.

Fluid(Particles):

- Not compressional: $\vec{k} \cdot \vec{U}_{s1} = 0$, hence fluid pressure and temperature do not come into play.
- The dominate fluid velocity is the ExB drift velocity \vec{E}_1/B_0 , but the total current \vec{J}_1 only has contributions from the polarization drifts.
- Ions and electrons are moving together with the velocity $\pm E_1/B_0$.

EM Wave and Fluid:

- The total current introduces $\vec{J}_1 \times \vec{B}_0$ force that is $\pi/2$ out of phase of ExB drift velocity, causing particles moving up and down.
- Particles are moving together with the magnetic field line like they are frozen to the lines.
- Hence, Alfvén wave is the hydromagnetic wave that are coupled with both hydrodynamic (fluid) and electromagnetic phenomena.

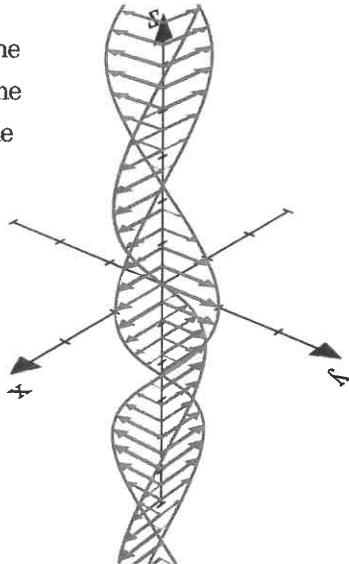
VIII Cartoon

Qualitatively, \vec{E}_1 , \vec{B}_1 , and total current \vec{J}_1 :

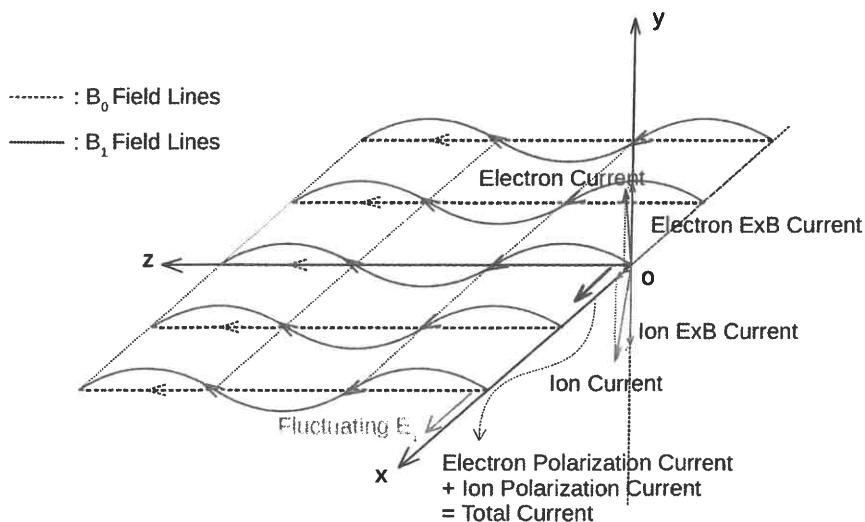
Electric Field: Orange Line

Magnetic Field: Green Line

Total Current: Purple Line



Qualitatively, the sinusoidal ripple of the magnetic field lines:



Reference

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- Stix, Thomas H. *Waves in plasmas*. Springer Science & Business Media, 1992, Section 2-4.
- Hazeltine, Richard D., and François L. Waelbroeck. *The framework of plasma physics*. CRC Press, 2018, Section 4.8
- Gurnett, Donald A., and Amitava Bhattacharjee. *Introduction to plasma physics: With space, laboratory and astrophysical applications*. Cambridge University Press, 2017, Section 3.6, 6.5.1
- UTAustin Plasma Physics Notes: <https://tarsige.ph.utexas.edu/teaching/plasma/lectures/node50.html#e4.57>
- Typical Alfvén wave animation: https://www.youtube.com/watch?v=7RB_kD9aSqc

Jacob Fruchtmann

[cold plasma] (fast) Magnetoacoustic wave

Quick reminder: Hot plasma fast wave
is of the form

$$\frac{\omega^2}{k^2} = \frac{1}{2} \left[C_S^2 + V_A^2 + \sqrt{(C_S^2 + V_A^2)^2 - 4 C_S^2 V_A^2 \sin^2 \theta} \right]$$

In cold plasma limit, $T \rightarrow 0 \Rightarrow C_S \rightarrow 0$

so have $\frac{\omega^2}{k^2} \rightarrow \frac{1}{2} [V_A^2 + \sqrt{V_A^4}] = V_A^2$

A. Limits of the Magnetog acoustic Wave ($k \perp B_0$)

a. $\omega \ll \omega_{BH}$

1. Take $\underline{k} = k \hat{x}, \underline{B}_0 = B_0 \hat{z}$

2. X-Mode has $\underline{E}_1 = (E_0, -\frac{i\omega}{D} E_0, 0)$

a. We can reparameterize this as $\underline{E}_1 = (\frac{iD}{\omega} E_0, E_0, 0)$
this will be useful for later.

3. Assume $q_i = q = e, n_i = n_e = N_0$

4. $\omega_H^2 = \omega_{ci}/\omega_{ce}$, Assuming $\omega_{pi} \gg \omega_{ci}/\omega_{ce}$

5. We will investigate the wave behavior in limit $k \rightarrow 0$

a. Note from Lecture 3 (C.1) (Page 3):

As $k \rightarrow 0$ for X-Mode curve, $\omega \rightarrow 0$.

This is the magnetosonic limit

b. Rigorously, only demand $\omega^2 \ll \omega_{ci}\omega_{ce} \ll \omega_{ce}^2$
not $\omega \ll \omega_{ci}$. Not demanding this turns

(ω, k) into a mess to solve. ~~mess~~

B. Ion & electric current:

$$1. \quad U_s = i \frac{q_s}{\omega_m} E_1 + i \frac{\omega_{cs}}{\omega} U_s \times \hat{b}$$

(1) (2) (3)

2. Comparing terms, we have

$$\lim_{t \rightarrow 0} \frac{(3)}{(1)} = \lim_{t \rightarrow 0} \frac{\omega_{cs}}{\omega} = \lim_{\omega \rightarrow 0} \frac{\omega_{cs}}{\omega} = \infty$$

3. Lorentz Force dominates, charged particles are magnetized!

4. Larmor Motion is lowest order motion.

5. From single particle drifts:

$$a. \quad U_s = \frac{i q_s}{\omega_m} E_z \hat{b} + \frac{E_x b}{B_0} - \frac{i \omega}{\omega_m B_0} E_1$$

$$b. \quad j_s = n_s q_s U_s = n_s q_s \frac{E_x b}{B_0} - i \omega \left(\frac{q_s}{m_s} \right) \left(\frac{n m_s}{B_0^2} \right) E_1$$

$$c. \quad \mu_0 j_s = \mu_0 n_s q_s \frac{E_x b}{B_0} - i \omega \left(\frac{\mu_0 \rho_s}{B_0^2} \right) E_1$$

$$d. \quad \boxed{\mu_0 j_s = \mu_0 n_s q_s \frac{E_x b}{B_0} - i \omega \frac{1}{V_A^2} E_1}$$

$$6. \quad \text{Ion current: } \boxed{\mu_0 j_i = \mu_0 n_e \frac{E_x b}{B_0} - i \omega \frac{1}{V_{Ai}^2} E_1}$$

$$7. \quad \text{Electron current: } \boxed{\mu_0 j_e = -\mu_0 n_e \frac{E_x b}{B_0} - i \omega \frac{1}{V_{Ae}^2} E_1}$$

$$8. \quad \text{Total current: } \boxed{\mu_0 j = -i \omega \left[\frac{1}{V_{Ai}^2} + \frac{1}{V_{Ae}^2} \right] E_1 \approx -i \omega \frac{1}{V_A^2} E_1}$$

$$(V_{Ai}^2 \ll V_{Ae}^2)$$

C. Limiting Behavior of Magnetoacoustic wave as $k \rightarrow 0, \omega \rightarrow$

1. Lowest order limit: $\tilde{E}_1 = \left(\frac{iD}{S} E_0, E_0, 0 \right)$

a. As $k \rightarrow 0, \omega \rightarrow 0 \Rightarrow D \rightarrow 0, S \rightarrow 1 + \sum \left(\frac{\omega s}{\omega s} \right)^2$

b. So $\lim_{k \rightarrow 0} \frac{\tilde{E}_x}{\tilde{E}_y} \rightarrow 0$

c. $\tilde{E}_{1(0)} = E_0 \hat{y}$

D. ~~Limiting behavior~~ Solution to the mode frequency

1. Faraday's Law: $B_1 = \frac{1}{\omega} \vec{k} \times \vec{E}_1 \quad (A)$

2. Ampere-Maxwell: $i \vec{k} \times \vec{B}_1 = \mu_0 \vec{j} - i \frac{\omega}{c^2} \vec{E}_1 \quad (B)$

3. Plugging (A) into (B), get

$$\tilde{\omega} \left[(\vec{k} \cdot \vec{E}_1) \vec{k} - k^2 \vec{E}_1 \right] = -\omega \left[\frac{1}{V_A^2} + \frac{1}{c_s^2} \right] \vec{E}_1 = -\omega \frac{1}{V_A^2} \left[1 + \frac{V_A^2}{c_s^2} \right] \vec{E}_1$$

4. In the non-relativistic limit, $V_A^2 \ll c^2$, this becomes:

$$V_A^2 \left[(\vec{k} \cdot \vec{E}_1) \vec{k} - k^2 \vec{E}_1 \right] + \omega^2 \vec{E}_1 = 0$$

$$\Rightarrow \left(\frac{\omega^2}{V_A^2} - \omega^2 - k^2 V_A^2 \right) \frac{(\vec{E}_x)}{(\vec{E}_y)} = 0$$

5. To lowest order, $\vec{E}_{1(0)} = E_0 \hat{y}$, this reduces to

$$(\omega^2 - k^2 V_A^2)(E_0) = 0$$

6. $\omega = \pm k V_A$

Magnetoacoustic

E. Physics of the wave

1. Current in the $\hat{k} \times \hat{x}$ direction

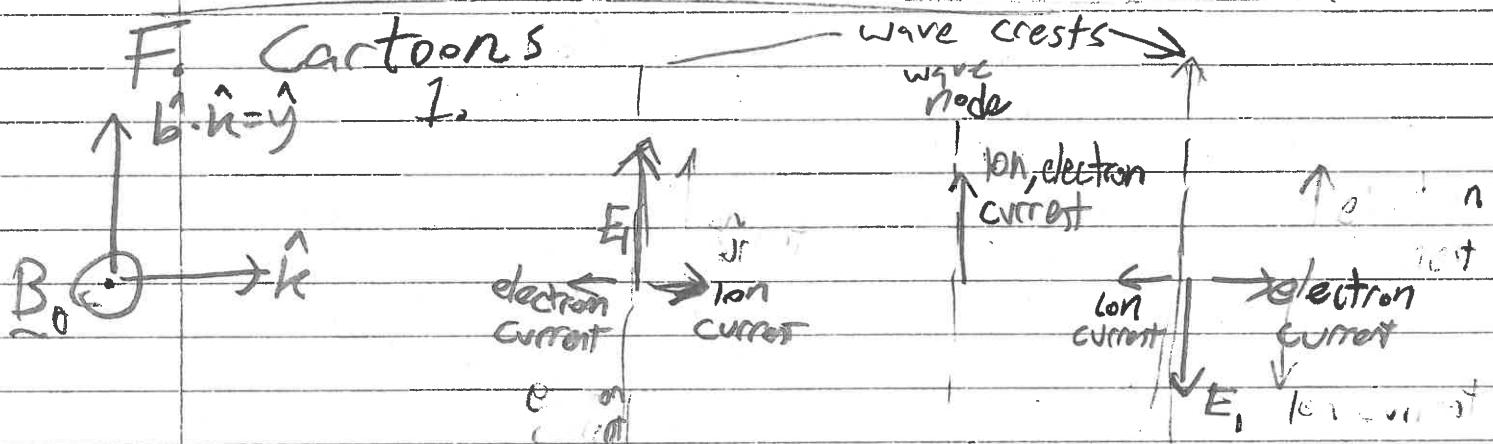
Sums to zero: This is from (i) ion $E \times B$ drift
 (ii) electron $E \times B$ drift

2. Non Zero current in E_y direction

From ion + electron polarization
 (+ displacement current)

3. $k \times B \neq 0, \omega B \neq 0$: supports B_z fluctuations

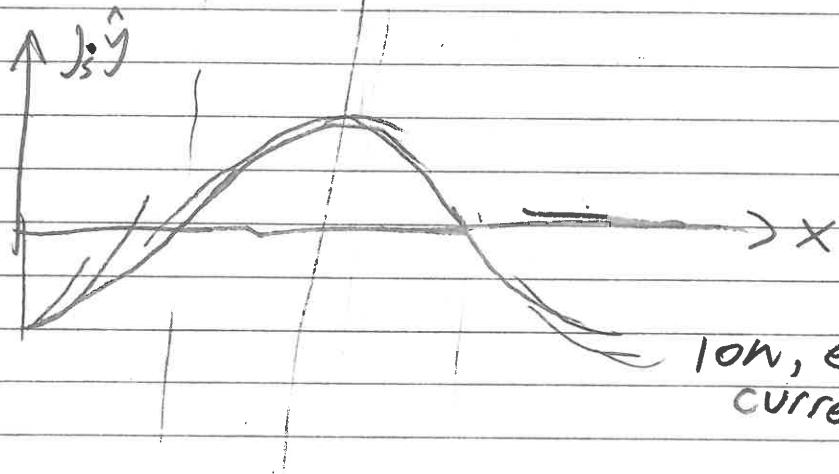
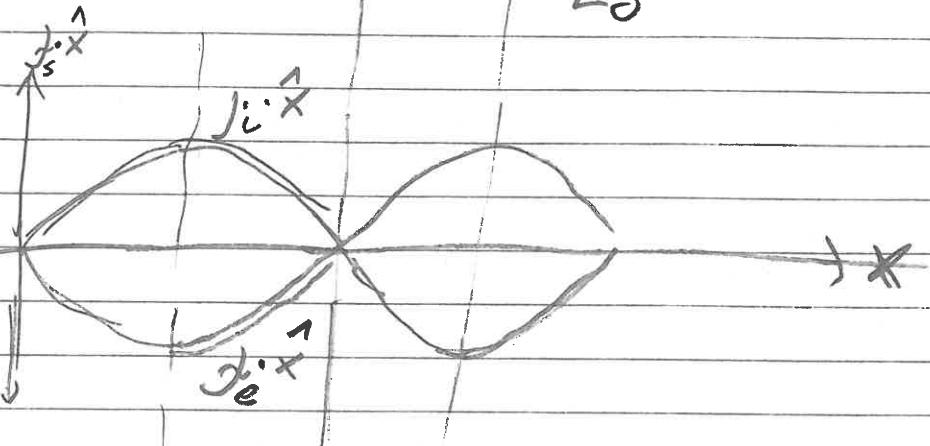
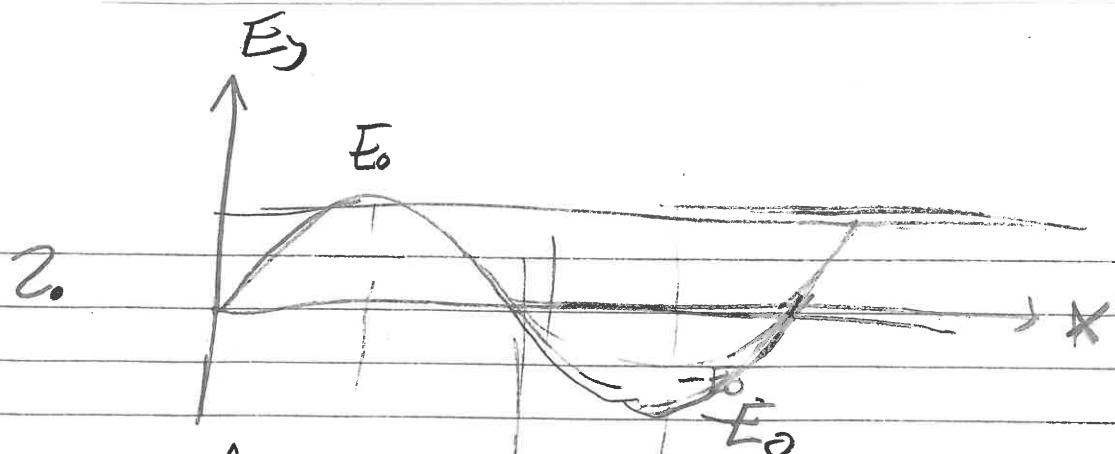
F. Cartoons



At timescales $\omega \ll \omega_{ci} \ll \omega_{ce}$, no charge build up

$$\nabla \cdot \vec{j}_e = 0$$

$$\vec{k} \cdot \vec{j}_e = 0$$



ion, electron
current

LH/RH Circularly Polarized Modified Light Waves

A. Limits of the Wave

For these wave modes

- \vec{k} and \vec{B}_0 are parallel and aligned along the z-axis so $\vec{k} = k\hat{z}$, $\vec{B}_0 = B_0\hat{z}$.

- the electric field is given by

$$\vec{E}_1 = E_1 \begin{pmatrix} +i \\ 1 \end{pmatrix}$$

eq1

with $E_z = 0$, an implicit $e^{ik(z-wt)}$ dependence, and the $\mp i$ term introducing a phase difference between the E_x, E_y terms which makes the mode right-hand/left-hand circularly polarized - the case of $w > w_L \gg |w_{ce}| \gg w_c$

B. Ion and Electron Current

The momentum equation gives

$$m_s n_s (\partial \vec{U}_s / \partial t + \vec{U}_s \cdot \nabla \vec{U}_s) = q_s n_s (\vec{E}_1 + \vec{U}_s \times \vec{B})$$

eq2

After linearizing the different quantities

$$-im_s n_{s0} w \vec{U}_{s1} = q_s n_{s0} (\vec{E}_1 + \vec{U}_{s1} \times \vec{B}_0)$$

eq3

$$\vec{U}_{s1} = \frac{q_s}{m_s} \frac{1}{w_{cs}^2 - w^2} \begin{pmatrix} -iw & w_{cs} \\ -w_{cs} & -iw \end{pmatrix} \vec{E}_1$$

eq4

$$\vec{U}_{s1} = U_{s1} \begin{pmatrix} +i \\ 1 \end{pmatrix}$$

eq5

$$U_{s1} = -i \frac{q_s}{m_s} \frac{w}{w_{cs}^2 - w^2} E_1 \approx i \frac{q_s}{m_s w} E_1$$

eq6

The current associated with each species is given by

$$\vec{J}_s = n_s q_s \vec{U}_s$$

eq7

Once again, after linearization

$$\vec{J}_s = n_{so} q_s \vec{U}_{si}$$

eq8

$$\vec{J}_s = \vec{J}_s \begin{pmatrix} +i \\ 1 \end{pmatrix}$$

eq9

$$\vec{J}_s = i \frac{n_{so} q_s^2}{m_s} \frac{\omega}{\omega^2 - \omega_{cs}^2} \vec{E}_1 \approx i \frac{n_{so} q_s^2}{m_s} \frac{1}{\omega} \vec{E}_1$$

eq10

The same form as \vec{E}_1 , but out of phase by $\pi/2$ due to the factor of i in eq10.

C. Limiting Behavior / Solution of the Mode Frequency

The general dispersion relation for these modes is given by

$$\frac{\omega^2}{k^2} = c^2 \frac{(w + |w_{ce}|)(w + w_{ci})}{(w + w_L)(w + w_R)} \quad \text{eq11}$$

In the limit $w > w_L \gg |w_{ce}| \gg w_{ci}$

$$\frac{\omega^2}{k^2} \approx c^2 \frac{w^2}{(w + w_L)(w + w_R)} \quad \text{eq12}$$

$$\frac{k^2 c^2}{\omega^2} \approx w^2 + w(w_{ce} + w_{ci}) - (w_p^2 - w_{ce} w_{ci}) \quad \text{eq13}$$

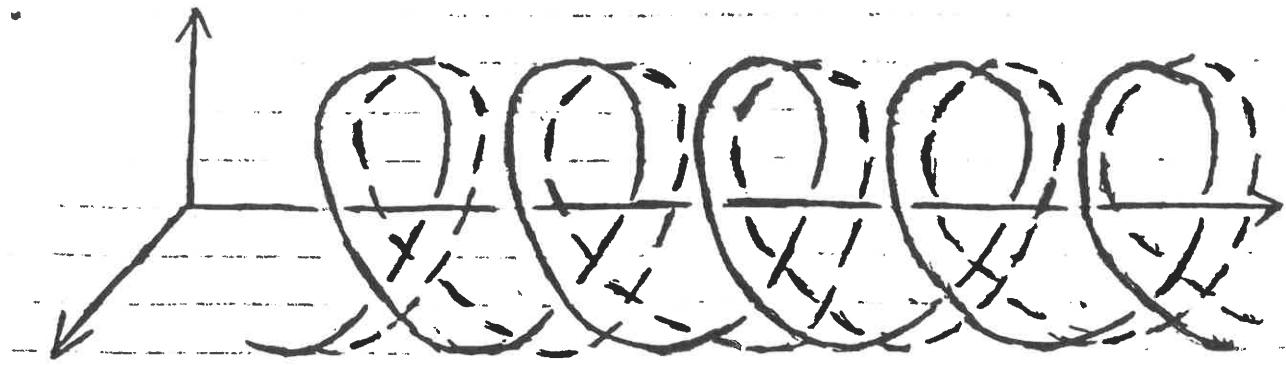
$$\frac{k^2 c^2}{\omega^2} \approx w^2 - w_p^2 \quad \text{eq14}$$

The same basic form as the dispersion relation for standard transverse light waves but with an additional $-w_p^2$ term.

D. Physical Description of the Mode

eq1, eq9, eq10

Given the forms of \vec{E} and \vec{J} , both are circularly polarized waves with the same handedness which are transverse to their directions of propagation, but with \vec{J} offset from \vec{E} by a phase of $\pi/2$.



E. Faraday Rotation

A linearly polarized wave can be expressed as the superposition of a LH circularly polarized mode and a right RH circularly polarized mode, however since these modes have different dispersion relations and different velocities, the phase between the two changes with z , causing the polarization to rotate but remain linear. Consider

$$\vec{E} = (\vec{E}_R/2) + (\vec{E}_L/2)$$

$$\vec{E} = \frac{E_i}{2} \left(\cos(k_L z - \omega t) + \cos(k_R z - \omega t) \right) \quad \text{eq15}$$

$$\vec{E} = \frac{E_i}{2} \left(\sin(k_L z - \omega t) - \sin(k_R z - \omega t) \right) \quad \text{eq16}$$

$$\vec{E} = E_i \left(\cos(((k_L - k_R)/2)z) \cos(((k_L + k_R)/2)z - \omega t) \right) \quad \text{eq17}$$

The angle of polarization can be expressed as

$$\tan \psi = E_y / E_x \quad \text{eq18}$$

$$\tan \psi = \tan((k_L - k_R)/2)z \quad \text{eq19}$$

$$\psi = \frac{k_L - k_R}{2} z \quad \text{eq20}$$

the dispersion relation eq11 can also be expressed as

$$k^2 = \frac{\omega^2}{c^2} \left(1 - \sum_s \frac{w_p^2}{\omega(\omega + w_s)} \right) \quad \text{eq21}$$

Given eq20, eq21

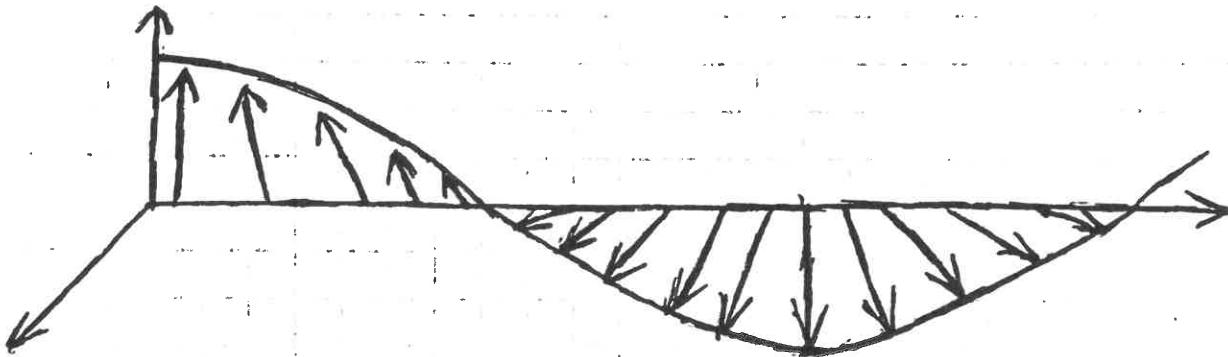
$$\Psi = \frac{1}{2} \frac{\omega}{c} \left(\left(1 - \sum_s \frac{\omega_{ps}^2}{\omega(\omega + \omega_{ps})} \right)^{1/2} - \left(1 - \sum_s \frac{\omega_{pi}^2}{\omega(\omega - \omega_{ps})} \right)^{1/2} \right) z \quad e922$$

Given $\omega \gg \omega_{ps} - \omega_{pi}$

$$\Psi \approx \frac{1}{2} \frac{\omega}{c} \left(\left(1 - \frac{1}{2} \sum_s \frac{\omega_{pi}^2}{\omega^2} \left(1 - \frac{\omega_{ps}}{\omega} \right) \right)^{1/2} - \left(1 - \frac{1}{2} \sum_s \frac{\omega_{pi}^2}{\omega^2} \left(1 + \frac{\omega_{ps}}{\omega} \right) \right)^{1/2} \right) z \quad e923$$

$$\Psi \approx \frac{1}{2c} \frac{1}{\omega} \left(\omega_{pi}^2 w_{ci} + \omega_{pe}^2 w_{ce} \right) z \quad e924$$

And so the angle of polarization changes with z , meaning the linearly polarized wave rotates over z , with the rate of rotation dependent on B_0, n_0 .



This calculation can be generalized to cases where n_0, B_0 vary with z to similar results. Due to the dependence of the rate of change of Ψ on B_0, n_0 waves such as these can be used to measure particle density or field strength.