Lecture #4  Boldyrev's Theory for Strong MHD Turbulence  

I. Review of GS's Strong MHD Turbulence Theory  

A. Assumptions  
1. Keldysh's Hypothesis: a. Local energy transfer  
   b. Constant energy cascade rate  
2. Anisotropic Cascade  
   Nonlinear turbulence frequency determined by perpendicular dynamics, \( \omega \sim k_L v_k \)  
3. Critical Balance between linear and nonlinear timescales, \( \omega \sim \omega_n \)  

B. Predictions:  
1. \( \omega_n \sim k_L v_k \)  
   [where \( v_k = \sqrt[3]{(k)} \)]  
2. \( v_k = C_0^{-\frac{1}{3}} k_L^{-\frac{1}{3}} \)  
3. \( E_{k_L} = C_0^{-\frac{2}{3}} k_L^{-\frac{5}{3}} \), Goldreich-Sridhar spectrum \( \propto k_L^{-\frac{5}{3}} \)  
4. \( k_{||} = C_0^{\frac{1}{3}} k_L^{-\frac{2}{3}} \), Scale-dependent anisotropy, \( k_{||} \propto k_L^{\frac{2}{3}} \)  

C. NOTE:  
1. Isotropic dynamics in perpendicular plane  
   \[ \begin{array}{c} \hat{\mathbf{k}}_L \\ \hat{\mathbf{l}}_L \end{array} \]  
   \[ \begin{array}{c} \hat{\mathbf{E}}_L \\ \hat{\mathbf{B}} \end{array} \]  

2. Thus, turbulent structures at small scales (with \( k_L \gg k_{||} \)) are filamentary, or cigar-shaped, with elongation along the mean magnetic field.  
3. The Boldyrev theory finds anisotropy in the perpendicular plane.
II. Boldyrev's Theory
   A. Motivation:
   1. Although the GS95 theory for strong MHD turbulence accomplished a great stride forward in our fundamental understanding and the inevitability of anisotropy, detailed numerical simulations of MHD turbulence cannot specify that appeared to scale like \( k^{-\frac{5}{3}} \) instead of \( k^{-\frac{5}{3}} \).
      (We'll review these simulation results in a later lecture.)
   2. This spectrum disagreed with both:
      a. Goldreich-Sridhar 95, \( E(k) \propto k^{-\frac{5}{3}} \), anisotropic
      b. Vishniakov-Konchashvili, \( E(k) \propto k^{-\frac{3}{2}} \), isotropic.
   3. Studies of decaying MHD turbulence (as opposed to driven turbulence) find a tendency towards dynamic alignment, where the fluctuations approach the sense of either:
      \[ \nabla \psi = b \perp (x) \quad \text{or} \quad \nabla (x) = -b \perp (x) \n\]
   a. \( \psi^+ = 0, \quad \psi^- = 0 \) or \( \psi^+ = 0, \quad \psi^- \neq 0 \)
   b. Nonlinear interaction is zero in either of these states!
      \( (\psi^+, \nabla) \psi^+ \) or \( (\psi^-, \nabla) \psi^- \).
   4. The Boldyrev theory proposes that this tendency to approach dynamic alignment occurs also in driven MHD turbulence, but the turbulence may only achieve an imperfect (and scale dependent) alignment while maintaining a constant energy flux to small scales.
B. Geometry of Nonlinear Interactions:

1. Equal Amplitude, Perpendicularly Polarized Alfvén Wavepacket Collisions:
   a. \( + \) wave: \( z^+ = z^+ \hat{y} \) and \( k_1^+ = k_1^+ \hat{y} \)
   \( \Theta \) wave: \( z^- = z^- \hat{y} \) and \( k_1^- = -k_1^- \hat{y} \)

   (not to scale)

2. Unequal Amplitude, Perpendicularly Polarized Alfvén Wavepackets:

3. Equal Amplitude, Non-perpendicular Polarized Alfvén Wavepackets
II. (Continued)

C. Bolotnaya's Approach:

1a. Assume that, for fluctuations at some perpendicular scale \( k_1 \), magnetic & velocity field fluctuations are aligned by some angle \( \theta_k \)

\[
\begin{align*}
\overline{z^+} &= \overline{v}_k + \overline{b}_1 \\
\overline{z^-} &= \overline{v}_k - \overline{b}_1
\end{align*}
\]


b. Thus \( |\overline{z^+}| = 2 \overline{v}_k \cos \left( \frac{\theta_k}{2} \right) \)

and \( |\overline{z^-}| = 2 \overline{v}_k \sin \left( \frac{\theta_k}{2} \right) \)

\[
\frac{\sin \theta}{\theta} \approx \begin{array}{|c|c|}
\hline
\theta & \sin \theta \\
1 & 0.8415 \\
0.707 & 0.6342 \\
0.5 & 0.4794 \\
0.383 & 0.393 \\
\hline
\end{array}
\]

2. The nonlinear term is \( (z^+ \cdot \nabla) z^+ \):

a. Thus \( \text{dropping the 2, } (z^+ \cdot \nabla) z^+ \sim \overline{v}_k^2 k_1^+ \theta_k \)

b. Equivalently, \( c_{\text{ne}} \sim |z^+| \sim k_1^+ \overline{v}_k \theta_k \)

2c. Since we have assume local interactions in scale-space, \( k_1^+ \sim k_1^- \sim k_1 \)

So we have \( c_{\text{ne}} \sim k_1^+ \overline{v}_k \theta_k \)

\[\text{NOTE: This differs from OS95 by the factor } \theta_k \ll 1.\]
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J. C. (Continued)

3. Assume Constant Energy Cascade Rate:
   a. \( E \sim \frac{V_k^2}{\nu_k} a \sim V_k^2 k_1 \theta_k = \tilde{E}_0 \)
   
   b. \( V_k = \tilde{E}_0^{1/3} \theta_k^{-1/3} k_1^{-1/3} \)
      Thus \( V_k \sim \theta_k^{-1/3} k_1^{-1/3} \)

4. Critical Balance: Linear ~ Nonlinear Frequencies
   \( \omega \sim \omega_{ne} \)
   
   a. \( \omega = k_{ll} V_A \)
      \( \omega_{ne} \sim k_1 V_k \theta_k \)
      \( \Rightarrow k_{ll} V_A \sim k_1 V_k \theta_k \sim k_1 (\tilde{E}_0^{1/3} \theta_k^{-1/3} k_1^{-1/3}) \theta_k \)
   
   b. Thus \( k_{ll} = \frac{\tilde{E}_0^{1/3}}{V_A} \theta_k \theta_k^{1/3} k_1^{1/3} \)
      \( k_{ll} \sim \theta_k \theta_k^{1/3} k_1^{1/3} \)

5. Structure in the \( B_0 \times k \) direction:
   a. Consider the \( z^+ \) wave packet
      \[
      \text{Variation in } z \text{ direction } \sim \frac{i}{k_z}
      \]

   b. What is the variation in the \( y \) direction?
      \[
      \text{Consider the displacement of the magnetic field lines in } y.
      \]
      \[
      \frac{B_0 \sim \xi_i}{l_1 \sim k_1} \quad \Rightarrow \quad \frac{V_k \sim k_{ll}}{V_A \sim k_1} \quad \text{Thus} \quad \frac{k_{ll}}{k_1} \sim \frac{E_0^{1/3} \theta_k^{-1/3} k_1^{-1/3}}{V_A}.
      \]
Lemma 24

II. C. 5. b. (Continued)

\[ \frac{E \Theta}{V_k^3 \theta_k^3 k_L^3} \sim \frac{E \Theta}{V_k^3 \theta_k^3 k_L^3} \Rightarrow k_i \sim k_L \theta_k \]

c. Thus, we find \( \frac{k_{ii}}{k_i} \sim \frac{v_k}{v_L} \ll 1 \) and \( \frac{k_{ii}}{k_L} \sim \theta_k \ll 1 \), so

\( k_{ii} \ll k_i \ll k_L \)

\[ k_i \sim \frac{1}{k_i} \]

\[ k_L \sim \frac{1}{k_L} \]

6. Assume All Quantities are Scale Invariant (including \( \Theta_k \))

a. Take \( \Theta_k \propto k_L^{-3} \frac{\alpha}{3 \pi x} \)

b. Determine scaling of \( v_k, k_i, \) and \( k_{ii} \) in terms of \( k_L \) & \( \alpha \):

1. \( v_k \propto \Theta_k^{-\frac{1}{3}} k_L^{-\frac{1}{3}} \propto \left[ k_L^{\frac{3}{35}} \right] \Rightarrow v_k \propto k_L^{-\frac{1}{35}} \)

2. \( k_i \propto k_L \theta_k \propto k_L \frac{3}{35} \alpha \frac{x}{3 \pi x} \propto k_L \frac{3}{35} \Rightarrow k_i \propto k_L \frac{3}{35} \)

3. \( k_{ii} \propto \Theta_k^\frac{2}{3} k_L^\frac{2}{3} \propto \left[ k_L^{\frac{2}{35}} \frac{2 \alpha}{4 \pi x + 3} \right] \propto k_L^{\frac{2}{35}} \Rightarrow k_{ii} \propto k_L^{\frac{2}{35}} \)
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II. C. 6. (Confinement)

1. Thus, we have defined a one-parameter family of solutions:

\[ \Theta_k \propto k_1 \frac{x}{3 \alpha}, \quad v_k \propto k_1 \frac{1}{3 \alpha}, \quad k_1 \propto k_1 \frac{3}{3 \alpha}, \quad k_{11} \propto k_1 \frac{2}{3 \alpha} \]

2. However, \( \alpha \) remains undetermined thus far.

3. \[ \text{NOTE: } \alpha = 0 \text{ corresponds to GS95 theory} \]

\[ \Theta_k = \text{constant}, \quad v_k \propto k_1 \frac{1}{3}, \quad k_1 \propto k_1 \frac{3}{3}, \quad k_{11} \propto k_1 \frac{2}{3} \]

\[ \text{isotropic in perpendicular plane} \]

7. Conservation of Cross Helicity

a. In incompressible MHD, \( H_c \equiv \int d^3x \frac{1}{2} \mathbf{V} \cdot \mathbf{b} \)

cross helicity is conserved.

b. We will choose \( \alpha \) such that cross helicity is maximized. This is motivated by decaying MHD turbulence, in which the turbulence approaches a maximally aligned state, \( \mathbf{V}(1) = \mathbf{b}(1) \), or \( \mathbf{V}(1) = -\mathbf{b}(1) \).

c. This concept of dynamic alignment is the new physical phenomenon distinguishing Balakirev's theory from GS95.

d. We want maximal alignment (minimum of angular mismatch) between \( \mathbf{v}_k \) and \( \mathbf{b}_k \) as \( \alpha \) is varied.

e. In the perpendicular plane, \[ \Theta_k \propto k_1 \frac{x}{3 \alpha}, \quad v_k \propto \frac{\mathbf{b}_k}{\Theta_k}, \quad \mathbf{b}_k \propto \frac{\mathbf{b}_k}{\Theta_k} \]

So \( \alpha \to \infty \) leads to a minimum of \( \Theta_k \).
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II. b) (Concluded)  
F: BUT, $v_k$ & $b_k$ are also mismatched out of the perpendicular plane. 

\[ l_k = \frac{1}{k_k} \]
\[ l_1 \approx \frac{1}{k_1} \]

Angle between magnetic field direction within perpendicular scale $l_\perp \sim \frac{1}{k_\perp}$. 

$\Rightarrow$ The direction of the local magnetic field at scale $l_\perp \frac{1}{k_\perp}$ cannot be defined precisely. 

1. Again 
\[ \frac{\beta_k}{k_\perp} \sim \frac{\beta_k}{k_k} \sim \frac{\beta_k}{l_\perp} \sim \tan \frac{\beta_k}{2} \]

2. Taking $\beta_k \ll 1$, $\tan \beta_k \sim \beta_k$, so $\frac{l_k}{l_\perp} \sim \frac{k_k}{k_\perp} \sim \frac{1}{3k_\perp}$ 

3. So $\beta_k \sim \frac{k_\perp}{k_k} \sim \frac{k_\perp}{k_k \frac{1}{3k_\perp}} \sim k_\perp$ 

4. The total angle is $\Theta_k = \sqrt{\Theta_k^2 + \beta_k^2}$.

This angle is minimized, with respect to $\alpha$, when $\alpha = 1$ ($\Theta_k = \beta_k$).  $\Rightarrow$ $\alpha = 1$

b. Scalings: 
- $\Theta_k \propto k_\perp^{-\frac{1}{2}}$ 
- $v_k \propto k_\perp^{-\frac{1}{2}}$ 
- $k_i \propto k_\perp^{-\frac{3}{4}}$ 
- $k_{\perp} \propto k_i^{-\frac{1}{2}}$
- $E_k \propto \frac{V_k^2}{k_\perp} \propto k_\perp^{-\frac{3}{2}}$

$E_{k_i} \propto k_i^{-\frac{3}{2}}$ Boldly new spectrum
II. (Continued)

Dr. Predictions of Boldyrev Theory compared to GS95

1. Turbulence is essentially 3-dimensional, with anisotropy between all axes.

\[ \begin{align*}
\text{GS95:} & \quad l_\parallel \sim \frac{1}{k_1} \\
\text{Boldyrev:} & \quad l_\parallel \sim \frac{1}{k_k}
\end{align*} \]

\[ k_\parallel \ll k_1 \ll k_2 \]

a. This leads to Boldyrev theory leads to current sheets at small scales in MHD turbulence, consistent with simulations.

b. GS95 predicts small-scale filaments not observed.

2. \[ \frac{(k_{1,0})}{(k_{1,0})} \]

\[ \begin{align*}
\text{Development of perpendicular anisotropy} & \quad (\text{due to dynamic alignment}) \\
\text{GS95:} & \quad k_{1,0} \propto k_1^{2/3} \\
\text{Boldyrev:} & \quad \text{Stronger anisotropy than GS95}
\end{align*} \]

a. Take isotropic driving \( k_\parallel = k_1 = k_2 = K_0 \) with \( V_0 = V_A \Rightarrow \theta_0 = 1 \)

b. scalings: \( \theta_k = \theta_0 \left( \frac{k_1}{K_0} \right)^{-2/3} \), \( V_k = V_A \left( \frac{k_1}{K_0} \right)^{-2/3} \), \( k_\parallel = \theta_0 K_0^{1/3} k_1^{1/6} \), \( k_i = \theta_0 K_0^{1/3} k_1^{1/6} \)
E. Parallel Spectrum:

1. We can determine the energy spectrum in $k_{\parallel}$ by using
   \[ E = \int_0^{\infty} dk_{\perp} E(k_{\perp}) = \int_0^{\infty} dk_{\parallel} E(k_{\parallel}) = 2 \int_0^{\infty} dk_{\parallel} E(k_{\parallel}) \]

   a. Thus, \[ E(k_{\parallel}) = \frac{1}{2} E(k_{\perp}) \left( \frac{dk_{\parallel}}{dk_{\perp}} \right) \]

2. Boldyrev: \[ E_{\parallel} \propto k_{\perp}^{-2/3}, \quad k_{\parallel} \propto k_{\perp}^{1/2} \]
   a. \[ \frac{dk_{\parallel}}{dk_{\perp}} \propto \frac{1}{2} k_{\perp}^{-1/2} \]

   b. Thus, \[ E(k_{\parallel}) \propto \frac{\sqrt{2} k_{\perp}^{-2/3}}{\frac{1}{2} k_{\perp}^{-1/2}} \propto k_{\perp}^{-1} \propto k_{\parallel}^{-2} \]

   \[ \text{So, } E(k_{\parallel}) \propto k_{\parallel}^{-2} \quad \text{Boldyrev 1-0 parallel spectrum} \]

3. GS95: \[ E_{\parallel} \propto k_{\perp}^{-5/3}, \quad k_{\parallel} \propto k_{\perp}^{2/3} \]
   a. \[ \frac{dk_{\parallel}}{dk_{\perp}} = \frac{2}{3} k_{\perp}^{-1/3} \]

   b. \[ E(k_{\parallel}) \propto \frac{\sqrt{2} k_{\perp}^{-5/3}}{\frac{2}{3} k_{\perp}^{-1/3}} \propto k_{\perp}^{-4/3} \propto k_{\parallel}^{-2} \]

   \[ \text{So, } E(k_{\parallel}) \propto k_{\parallel}^{-2} \quad \text{GS95 1-D parallel spectrum.} \]

F. Physical Difference between GS95 and Boldyrev

1. Consider Critical Balance where $\Theta_k < 1$
   \[ k_{\parallel} v_A \sim k_{\parallel} v_k \Theta_k \]
   a. This means weakens NL interaction requiring a higher value of $k_{\parallel}$ to achieve critical balance (due to dynamic alignment).

   b. But $k_{\parallel}$ is a geometrical argument based on $S_{\parallel}, d, B_0$, so does not change.

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Howes
II. (Continued)

2. Thus, it is dynamic alignment that leads to a thinning of the turbulent structures in the direction of $k_1$.

\[ G_{95} \rightarrow \quad B_{\text{Boliyrev}} \]

\[ k_{1G5} \Rightarrow \quad k_{1B} > k_{1G5} \]

\[ k_{1B} = k_{1G5} \quad (k_{1B} < k_{1G5}) \]

III. References:

   a. Best reference describing the theory and justifying physical arguments.

   a. Early version of the theory, with a subtly different geometry (uses II, B, 3, as the basis, rather than II, B, 2, as used in the 2006 paper). Not such compelling physical arguments.