Chap 3: The Heliocentric Model of the Solar System

The Eight Planets of the Solar System

Neptune
Uranus
Saturn
Jupiter
Mars
Earth
Venus
Mercury

1846
1781
1801
Ceres

Triton and more...
Ariel
Umbriel
Titania
Oberon and more...
Moons:
Enceinte
Tethys
Dione
Rhea
Titan
Hyperion that should have
Iapetus
and more...
Moons:
Europe
Ganymede
Callisto
and more...
Phobos
Deimos
Moon
Pluto
Chapter 3: The Heliocentric Model of the Solar System

- Heliocentric Model vs. Geocentric Model
  - *Retrograde motion* of Mars, Phases of Venus (Galileo)

- Geometric configurations of inner and outer planets
  - Conjunctions, Greatest Elongations, Quadratures

- Copernicus’ method of measuring *sidereal periods* of planets
  - From *Synodic Periods* of Planets and the Moon to *Sidereal Periods*

- Copernicus’ method of measuring *heliocentric distances* of planets
  - Inner planets (Greatest elongation)
  - Outer planets (time between quadrature and opposition)

- Kepler’s Laws (1610-1619 AD) [*Empirical Laws of Planetary Motion*]
  - Elliptical orbits, Equal area law, Period-Distance relation
  - Kepler’s triangulation method in determining Mars’ elliptical orbit
Part I: Geocentric vs. Heliocentric models
How did ancient Greek astronomers reject the hypothesis the Earth orbit around the sun?

- **Observation**: The Sun moves against background stars on the celestial sphere
- **Hypothesis**: The Earth moves around the Sun (which is much larger)
- **Experiment**: Measuring the expected motion of bright stars due to Earth’s orbital motion around the Sun (i.e., parallax)
- **Result**: no such yearly motion is detected for any bright stars
- **Updated Hypothesis**: The Earth is stationary and the Sun moves around the Earth
Aristotle’s Geocentric Model in 350 BC

Note: six of the eight planets known since 2000 BC, so this illustration is incomplete
Celestial Sphere is a Geocentric Point of View

The Sun constantly moves *eastwards* on the celestial sphere, causing the 4-min longer solar day than the sidereal day.

12:00:00  11:56:04  12:00:00

<table>
<thead>
<tr>
<th>23h 56′ 04″</th>
<th>3′ 56″</th>
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<tr>
<td>a sidereal day</td>
<td>a mean solar day</td>
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![Diagram showing the difference between sidereal and solar days](image-url)
But planets show \textit{retrograde} motions.
Ptolemy’s Geocentric Model in ~150 AD:
Only the Sun and the Moon show no retrograde motion

Think about this: how did they put the distances of the planets in order?
Aristarchus’ Heliocentric Model in ~250 BC

According to Archimedes’ book “*The Sand Reckoner*”:

You ["you" being the King] are now aware that the "universe" is the name given by most astronomers to the sphere the centre of which is the centre of the earth.

But Aristarchus has brought out a book consisting of certain hypotheses. His hypotheses are that the fixed stars and the sun remain unmoved, that the earth revolves about the sun on the circumference of a circle, the sun lying in the middle of the orbit.
1300 years after Ptolemy ... 

What a great hiatus! you don’t want to be a scientist living in that gap of progress
Resume: doctoral degree in canon law. A physician, a translator, a diplomat, a governor, and an economist. Lived in Poland.
Copernicus’s Heliocentric Model in 1540 CE
In a given time interval, the Earth will move farther in its orbit than the planet!

The planet is observed projected in a starry background.
How to explain retrograde motion of Mars in a Heliocentric Model? (Demo)

Why Mars appears the brightest during retrograde motion?
Inner planets (Venus and Mercury) also show retrograde motion like outer planets (e.g., Mars)
Which model is superior?

Ockham’s Razor (1300 CE)
but similar ideas from Aristotle and Ptolemy

“We may assume the superiority ceteris paribus [other things being equal] of the demonstration which derives from fewer postulates or hypotheses” - Aristotle

“With all things being equal, the simplest explanation tends to be the right one.” - Ockham
Which model is superior?

Although Ockham’s Razor (1300 CE) favors heliocentric model because it is much simpler.

A true experimental test of the two hypotheses is needed to decide which one is more correct.

To do this test, Astronomers would need better observations than simply recording the positions of planets as a function of time, which they had done for millenniums.
The utilization of telescopes marked the beginning of modern astronomy

Galileo Galilei (1564 - 1642 CE)

Following the invention of telescope in 1608 by eyeglass maker Lippershey, Galileo was the first person who used the tool in astronomy.

400 years later, we are still using exactly the same tool!
First, Galileo’s discovery of Jupiter’s largest moons showed that there are objects orbiting around Jupiter instead of the Earth.
Design an experiment with a high discrimination power

- **Geocentric Hypothesis:**
  Objects, including the Sun, orbit around the Earth

- **Prediction:** Viewed from Earth, Venus should never be more than half illuminated: i.e., no Gibbous phases

- **Heliocentric Hypothesis:**
  Objects, including the Earth, orbit around the Sun

- **Prediction:** Venus should show all phases, like the Moon

- **Experiment:** Telescope observations of Phases of Venus (Galileo 1610 CE)
Heliocentric Model: all phases of Venus - from crescent to full
With the help of his telescope, Galileo saw all phases of Venus in 1610 AD.
Notice how the angular size changes
Now in the 21st century, can you think of other observational methods to reject the geocentric model?
Transit of Venus (first observed in 1639 AD)
PARALLAX AND PARSECS
First parallax measurement of a distant star: 1838 CE by Bessel, 61 Cygni, \( \mu = 0.314 \) arcsec for a baseline of Earth’s orbit (1 Astronomical Unit)

proper motion of 61 Cygni A+B
Aberration of starlight (1727 AD by James Bradley)

\[
\theta - \phi \approx \tan(\theta - \phi) = \frac{v}{c} \approx 20''
\]
Geometric Configurations of Inner and Outer Planets:

These are special locations of planets on their orbit relative to Earth
**Elongation**: the angle between the Sun and the planet as seen from the Earth.
What is the phase of Venus at the greatest elongation?
Outer/Superior Planets: Opposition, Conjunction, Quadratures

What are the elongations at these four positions?
Configurations of Planets in Heliocentric Model
Part II: How Copernicus determined the heliocentric distances and sidereal periods of planets
Copernicus’ Method of Heliocentric Distances:

a. Inner Planets
Copernicus’ Heliocentric Distances vs. Modern Values

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Inner Planets: **Inferior/Superior Conjunctions, Greatest Elongations**

What is the phase of Venus at the greatest elongation?

Suppose you measure the greatest elongation of Venus to be 30 deg, what is the radius of its orbit around the Sun? Express in unit of A.U.
Copernicus’ method of determining the heliocentric distance of inner planets

Inner Planets: \( R_{\text{AU}} = \sin(\text{Max Elongation}) \)
For outer planets, the previous method no longer works, because the greatest elongation is the “opposition.”
Synodic Periods vs. Sidereal Periods

Synodic /səˈnædɪk/:
relating to or involving the conjunction of planets, moons, or other objects in the Solar system.

Origin: from Greek sunodikos, a meeting or an assembly
**Synodic Period**: The time that elapse between two successive identical configurations as seen from Earth. This can be easily measured by observers on Earth.

**Sidereal Period**: The time it takes the planet to complete one full orbit of the Sun. This cannot be directly measured for planets other than the Earth. How did Copernicus derive these periods?

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The table shows the measured and derived periods for each planet.
Copernicus’ Method of Determining Sidereal Periods of Planets
The Relation between **Synodic Period** & **Sidereal Periods**

**Key realization:** Between two successive oppositions, the faster planet has to travel *exactly one full circle* than the slower planet.

For uniform circular motions, the angle traveled in a period of \( t \) is \( t \) times angular speed \( \frac{360°}{P} \) (\( P \) the sidereal period), so the difference between the two planets is:

\[
t \times \frac{360°}{P_{\text{in}}} - t \times \frac{360°}{P_{\text{out}}}
\]

which is required to be a full circle if \( t \) is the synodic period \( S \):

\[
S \times \frac{360°}{P_{\text{in}}} - S \times \frac{360°}{P_{\text{out}}} = 360°
\]

\[
\Rightarrow \frac{1}{S} = \frac{1}{P_{\text{in}}} - \frac{1}{P_{\text{out}}}
\]
Use Synodic Periods to Determine Sidereal Periods of Planets

\[
\frac{1}{S} = \frac{1}{P_{\text{in}}} - \frac{1}{P_{\text{out}}}
\]

Synodic periods have been used to determine sidereal periods:

\[
\frac{1}{P_{\text{out}}} = \frac{1}{P_{\text{Earth}}} - \frac{1}{S}
\]

\[
\frac{1}{P_{\text{in}}} = \frac{1}{P_{\text{Earth}}} + \frac{1}{S}
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Synodic Period: The time that elapse between two successive identical configurations as seen from Earth. This can be easily measured by observers on Earth.

Sidereal Period: The time it takes the planet to complete one full orbit of the Sun. This cannot be directly measured for planets other than the Earth and is derived using the equations above.

\[
\frac{1}{P_{\text{out}}} = \frac{1}{P_{\text{Earth}}} - \frac{1}{S_{\text{out}}} \quad \& \quad \frac{1}{P_{\text{in}}} = \frac{1}{P_{\text{Earth}}} + \frac{1}{S_{\text{in}}}
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The **Synodic Period of a planet** is easy to measure, and we know the **sidereal period of the Earth**, so we can use the relation to calculate the **Sidereal Period of the planet** (which is difficult to measure directly):

\[
\frac{1}{S} = \frac{1}{P_{\text{in}}} - \frac{1}{P_{\text{out}}}
\]

If Venus has a synodic period of 484 days (as observed from the Earth), what is its sidereal period?

\[
S = 484 \text{ days} \\
P_{\text{out}} = 365 \text{ days} \\
\text{solve for } P_{\text{in}}
\]
The **Synodic Period of a planet** is easy to measure, and we know the sidereal period of the Earth, so we can use the relation to calculate the **Sidereal Period of the planet** (which is difficult to measure directly)

\[
\frac{1}{S} = \frac{1}{P_{\text{in}}} - \frac{1}{P_{\text{out}}}
\]

If Mars is observed to have a synodic period of 2.1 years, what is its sidereal period?

\[
S = 2.1 \text{ yrs} \\
Pin = 1.0 \text{ yr} \\
solve \text{ for } P_{\text{out}}
Copernicus’ Method of Heliocentric Distances:

b. Outer Planets
Copernicus’ method of determining the heliocentric distance of inner planets

**Inner Planets:** $R_{in} = \sin(\text{Max Elongation}) \text{ AU}$
Copernicus’ method of determining the heliocentric distances of outer planets

- the time between opposition and quadrature ($\Delta t$)
- outer planet’s synodic period ($S$)

\[
ESE' = 360^\circ \frac{\Delta t}{P_E}
\]
\[
PSP' = 360^\circ \frac{\Delta t}{P_P}
\]
\[
\alpha = ESE' - PSP'
\]
\[
= 360^\circ \Delta t \left( \frac{1}{P_E} - \frac{1}{P_P} \right)
\]
\[
= 2\pi \frac{\Delta t}{S}
\]
\[
R_{out} = SP' = \frac{1 \text{ AU}}{\cos \alpha}
\]
\[
= \sec \left( \frac{2\pi \Delta t}{S} \right) \text{ AU}
\]
Other examples: Synodic Period of the Moon - a Month
a lunar month is a **synodic period**: a full cycle of moon phases

orbital period is a **sidereal period**: time to orbit 360 degs around the Earth
Synodic Orbital Period of the Moon: the time between successive full Moons

\[ \frac{1}{S} = \frac{1}{P_{\text{in}}} - \frac{1}{P_{\text{out}}} \]

Given one lunar month is 29.5 days, what is the sidereal period of the Moon?

S = 29.5 days
\( P_{\text{out}} = 365 \) days
solve for \( P_{\text{in}} \)
Other Examples

Synodic Period of an Observer on Earth - *a Solar Day*
An observer on Earth is like a little moon orbiting around the Earth’s center once a day.
Synodic Spin Period of an observer on Earth: the time between successive noons (a Solar day)

\[ \frac{1}{S} = \frac{1}{P_{\text{in}}} - \frac{1}{P_{\text{out}}} \]

Given one solar day is 24 hrs, and a year is 365 days, what is the length of a sidereal day?

S = 24 hrs
P_{\text{out}} = 8760 \text{ hrs}
solve for P_{\text{in}}
Part III: Kepler’s Three Laws of Planetary Motion

Joannis Kepleri, Mathematici Carolinae, hanc Imaginem.
Argentoratensi Bibliothecae, Confect.
Matthias Berneggeri n.d. x m.m.
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Copernicus’ Heliocentric Distances

\[ R_{in} = \sin(\text{Max Elongation}) \text{ AU} \]

\[ R_{out} = \sec \left( \frac{2\pi \Delta t}{S} \right) \text{ AU} \]

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Copernicus’s Heliocentric Model in 1543 AD

Measured both the sidereal periods of planets (in years) and their heliocentric distances (in AU). What did he miss?
What did Copernicus miss?

- Unlike synodic period, sidereal period increases with heliocentric distance.
- There is an exact power-law relation between the two.

**TABLE 3.1 Distances of Planets from the Sun**

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Kepler’s 3rd law of Planetary Motion

\[ \frac{R^3}{P^2} = \text{constant related to the central object} \]
Kepler’s 3rd law of Planetary Motion

\[ \frac{R^3}{P^2} = \text{constant related to the central object} \]

Since Kepler found that planetary orbits are ellipses instead of circles, we replace the radius \( R \) above with the semimajor axis \( a \) below. And if we choose convenient units (AU & yr), the constant becomes unity:

Objects orbiting around the Sun: \( \frac{a_{AU}^3}{P_{\text{yr}}^2} = 1 \)

In the next Chapter, we will use Newton’s law of gravity to show the general form of Kepler’s 3rd law for planets in other star systems and for moons of planets:

Small objects orbiting around a much more massive object: \( \frac{a_{AU}^3}{P_{\text{yr}}^2} = M_{\text{solar-mass}} \)
Kepler’s 3rd Law: period-distance relation

\[ \frac{a^3}{P^2_{\text{year}}} = M_{\text{solar-mass}} \]

which is equivalent to:

\[ \left( \frac{a}{1 \text{ AU}} \right)^3 \left( \frac{P}{1 \text{ year}} \right)^{-2} = \frac{M}{M_{\text{sun}}} \]

Objects orbiting around the Sun

- Jupiter’s moons
- Saturn’s moons
- Eris
- Neptune
- Pluto
- Uranus
- Saturn
- Jupiter
- Ceres
- Mars
- Earth
- Venus
- Mercury
Kepler’s Laws of Planetary Motion: the 1st and the 2nd laws
Kepler’s 1st law of Planetary Motion

**Ellipse**: a regular oval shape, traced by a point moving in a plane so that the sum of its distances from the foci is constant, which equals 2x the semimajor axis \(a\)
Kepler’s 1st law of Planetary Motion

- Planet orbits are **ellipses**.
- The Sun is at one **focus** of the elliptical orbit.
Kepler’s 1st law of Planetary Motion

- Semimajor axis ($a$)
- Semiminor axis ($b$)
- Focal length ($F$)

Sun

Planet
The Eccentricity of Elliptical Orbits

The greater the eccentricity, the more elongated the ellipse.

- $e = 0.983$
- $e = 0.958$
- $e = 0.745$
- $e = 0$
The Eccentricity of Elliptical Orbits

- Focal Length $F$ is the distance between a focus and the center of the ellipse

- eccentricity:

$$e = \frac{F}{a} = \sqrt{1 - \frac{b^2}{a^2}}$$
Eccentricity of the planetary orbits

Mercury  0.206
Venus    0.007
Earth    0.017
Mars     0.093
Jupiter  0.048
Saturn   0.056
Uranus   0.046
Neptune  0.010
Pluto    0.248
Perihelion and Aphelion Distances

Aphelion distance:
\[ r_{ap} = a(1 + e) \]

Perihelion distance:
\[ r_{peri} = a(1 - e) \]
Kepler’s 2nd law of Planetary Motion

- Often called the **Law of Equal Areas**.
- The line between the Sun and the planet “sweeps” out **equal areas in equal times**.
Implications of Kepler’s 2nd law

A planet will go fastest when closest to the Sun, and it will go slowest when farthest from the Sun.

The area swept per unit time is $\frac{\Delta A}{\Delta t} = \frac{1}{2}r v_t$ (as shown below), and this quantity remains a constant throughout an orbit.

$$\Delta A \approx \frac{1}{2} r (v_t \Delta t) + \frac{1}{2} (v_r \Delta t)(v_t \Delta t) \approx \frac{1}{2} r (v_t \Delta t)$$
Physical Explanation of Kepler’s 2nd Law

• Angular momentum is preserved under the influence of a central force like gravity: $\vec{L} = \vec{r} \times m\vec{v} = mr v_t \vec{k}$ is a constant vector

• the product of heliocentric distance $r$ and the tangential velocity $v_t$ is a constant for each planet: $mr v_t = \text{constant}$

The 2nd law is equivalent to conservation of angular momentum

$$\Delta A \approx \frac{1}{2} r (v_t \Delta t) + \frac{1}{2} (v_r \Delta t)(v_t \Delta t) \approx \frac{1}{2} r (v_t \Delta t)$$
### Summary: Equations of Kepler’s Laws of Planetary Motion

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</tr>
<tr>
<td>[ r_{peri} = a(1 - e) ]</td>
<td>Perihelion distance</td>
</tr>
</tbody>
</table>

1st Law: elliptical orbits

2nd Law: equal area in equal time

3rd Law: period-distance relation

\[
\frac{a_{AU}^3}{P_{\text{year}}^2} = M_{\text{solar-mass}} \quad \text{which is equivalent to:}
\]

\[
\left( \frac{a}{1 \text{ AU}} \right)^3 \left( \frac{P}{1 \text{ year}} \right)^{-2} = \frac{M}{1 \text{ } M_{\text{sun}}}
\]
How did Kepler discover Mars’ elliptical orbit 400 years ago?

Yet another demonstration of how humans approach seemingly insurmountable problems with the tools we have at the time.
Tycho Brahe (1546-1601), Danish astronomer

For 20 years, he accumulated positional measurements of the five planets with an accuracy better than 1 arcmin (1/60°).
Uraniborg (dedicated to Urania, the Goddess of Astronomy)
Mural Quadrant in Tycho Brahe’s Observatory

mural quadrant: an altitude measurement instrument built into a wall on the meridian
In 1600, Kepler became Tycho’s assistant. After Tycho’s death in 1601, he used Tycho’s data to develop his three laws of planetary motion.
Kepler’s method of triangulating the distances to Mars

Choose two elongation measurements separated by a Martian year (687 days).

Why? Because Mars would have returned to its initial position \( M \), while Earth has moved from \( E_1 \) to \( E_2 \).

What did Kepler know?
1. The elongations, i.e., the angles \( SE_1M \) and \( SE_2M \)
2. The sidereal periods of the Earth and Mars
3. The time between the two measurements

What did Kepler want to solve?
The heliocentric distance \( MS \)
Kepler’s quadrilateral is fixed, because these measurements:
1. The elongations, i.e., the angles $SE_1M$ and $SE_2M$
2. The sidereal periods of the Earth and Mars
3. The time between the two measurements (687 days) tell us (a) three of the four angles and (b) two of its four sides.

To measure the heliocentric distance of Mars, simply draw the quadrilateral based on what we know, then measure $MS$ with a ruler.
Kepler’s Results on Mars

- Kepler did the triangulation calculations for five pairs of elongation data that are roughly evenly spaced along Mars’ orbit.

- He found the heliocentric distances of Mars vary quite a lot: \(~20\%\) difference for an eccentricity of 0.093.
Kepler’s method Step 1: Solve for \( E_1E_2 \)

*If* we assume uniform circular motion of the Earth, then \( E_1SE_2 \) is an isosceles triangle and we can calculate all its angles.

*How?* We just need to realize that Earth will return to \( E_1 \) in whole Earth years, and a Martian year is slightly shorter than 2 Earth years.

\[
E_1SE_2 = 360^\circ \frac{2P_E - P_M}{P_E}
\]

\[
E_1E_2 = 2 \sin \left( \frac{E_1SE_2}{2} \right)
\]
Kepler’s method Step 2: Solve for $E_2M$

**Focus on the triangle $E_1E_2M$**

We have solved the angle $SE_1E_2$, the angle $E_2E_1M$ is simply the elongation at $E_1$ minus $SE_1E_2$.

We also know $E_1E_2$ from Step 1.

Lastly, we know the angle $E_1ME_2 = 360 - ME_1S - ME_2S - E_1SE_2$.

Based on the law of sine, we can solve for $E_2M$:

$$\frac{E_2M}{\sin(E_2E_1M)} = \frac{E_1E_2}{\sin(E_1ME_2)}$$
Kepler’s method Step 3: Solve for MS

Focus on the triangle $SE_2M$

We know $SE_2=1$AU and $E_2M$ from Step 2

We also know the angle $SE_2M$ from the 2nd elongation measurement

Based on the law of cosine, we can finally solve for $MS$, the heliocentric distance of Mars:

$$MS^2 = SE_2^2 + E_2M^2 - 2 \cdot SE_2 \cdot E_2M \cdot \cos(SE_2M)$$