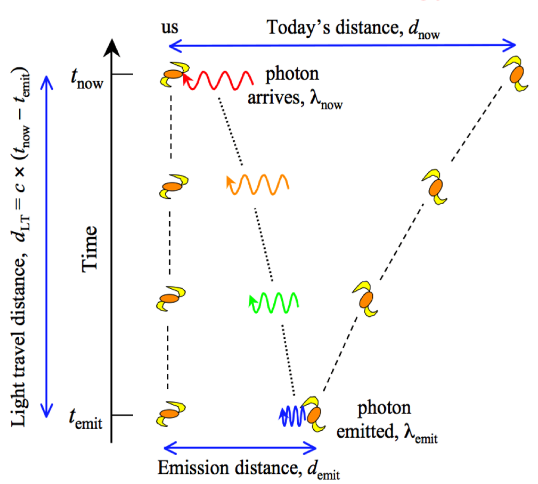
# Distances

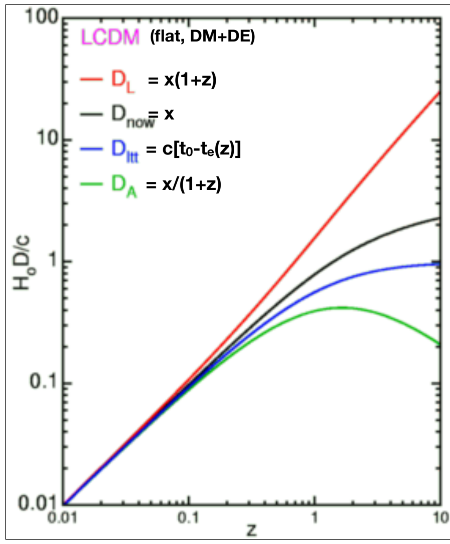
# Outline

- Spacetime Geometry
- Friedmann-Lemaitre-Robertson-Walker metric
- Cosmological redshift
- Distances
  - Comoving distance
  - Horizon distance
  - Proper distance
  - Luminosity distance
  - Angular diameter distance
  - Light travel time distance

## Distances in Cosmology

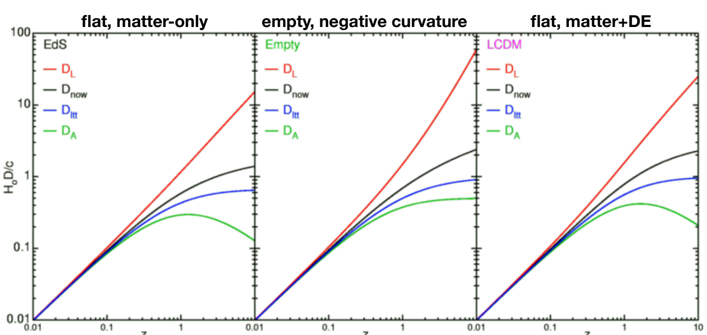



## D vs. z relation




Distance in unit of the Hubble scale: $D_H = c/H_0 = 3\ h^{-1}$ Gpc

## D(z) relations for different cosmologies

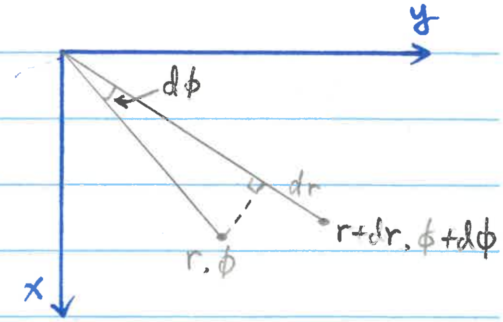



Distance in unit of the Hubble scale: $D_H = c/H_0 = 3\ h^{-1}$ Gpc

- Geometry
  - differential interval = metric = ds^2
  - spacetime interval = int sqrt(ds^2) along a worldline W
  - timelike (proper time), lightlike (frozen time), spacelike (proper distance)
- Friedmann-Lemaitre-Robertson-Walker metric
  - Introduce scale factor a(t) and comoving coordinates
  - Derivation of Hubble's law: v = dr/dt = x da/dt = r [da/dt/a]
- Cosmological redshift: (1+z) = 1/a(t)
- Distances
  - Comoving distance: x = c int dz/H(z)
    - Observer's horizon in comoving distance
  - Proper distance: D_p = a(t0) x = x for flat universe
    - Hubble law distance
  - Luminosity distance: D_L = (1+z) x
  - Angular diameter distance: D_A = x/(1+z) = emission distance
  - Light travel time distance: D_ltt = c (t_now-t_emi)

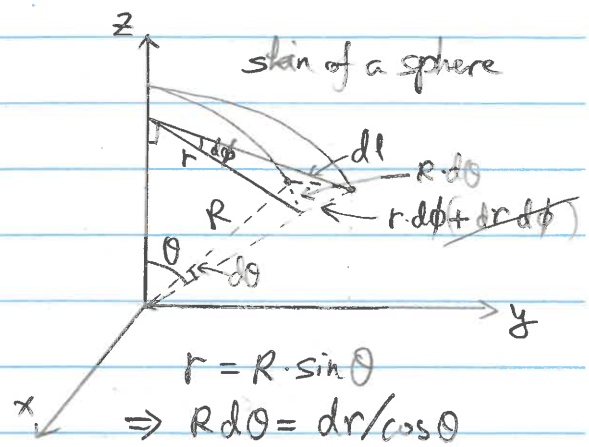
## Distances in curved space

example using polar coordinates in 2D




$$(d\ell)^2 = (dr)^2 + (r\,d\phi)^2 \quad , \quad \text{flat}$$

On a curved plane with curvature $K \equiv \frac{1}{R^2}$




$$r = R\cdot\sin\theta$$

$$\Rightarrow R\,d\theta = dr/\cos\theta$$

$$(d\ell)^2 = (R\,d\theta)^2 + (r\,d\phi)^2$$

$$= \left(\frac{dr}{\cos\theta}\right)^2 + (r\,d\phi)^2$$

$$= \left(\frac{dr}{\sqrt{1-r^2/R^2}}\right)^2 + (r\,d\phi)^2$$

$$= \left(\frac{dr}{\sqrt{1-Kr^2}}\right)^2 + (r\,d\phi)^2$$

extending to curved 3D space

$$(d\ell)^2 = (dr)^2 + (r\,d\theta)^2 + (r\sin\theta\,d\phi)^2 \quad , \quad \text{flat}$$

$$(d\ell)^2 = \left(\frac{dr}{\sqrt{1-Kr^2}}\right)^2 + (r\,d\theta)^2 + (r\sin\theta\,d\phi)^2 , \quad \text{curved}$$

space time metric for an isotropic, homogeneous, expanding universe

$$(ds)^2 \equiv (c\,dt)^2 - (d\ell)^2 \quad \leftarrow \text{defined to be invariant under Lorentz trans.}$$

$$= (c\,dt)^2 - \left(\frac{dr}{\sqrt{1-Kr^2}}\right)^2 - (r\,d\theta)^2 - (r\sin\theta\,d\phi)^2$$

define comoving radial coordinate $x \equiv r/a(t) \Rightarrow r = a(t)\cdot x$

& time independent curvature $k \equiv K\cdot a^2(t)$

$$\Rightarrow (ds)^2 = (c\,dt)^2 - a^2(t)\left[\left(\frac{dx}{\sqrt{1-kx^2}}\right)^2 + (x\,d\theta)^2 + (x\sin\theta\,d\phi)^2\right]$$

This is the Robertson-Walker metric.

Proper time : $\Delta\tau = \Delta s/c$ — time recorded by a watch moving along world line

Proper distance : $\Delta\ell = \sqrt{-(\Delta s)^2}$ — distance between two simultaneous events.

AG782 F21 : Cosmology p1

Lecture 1 outline:
redshift & distances

Lecture 2:
Friedmann Eq. & solutions

**Metrics:** differential spacetime interval

$$(ds)^2 = (c\,dt)^2 - (dl)^2 = (c\,dt)^2 - (dx)^2 - (dy)^2 - (dz)^2$$

total interval along a worldline $W$.

$$\Delta s = \int_A^B \sqrt{(ds)^2} \text{ along } W \qquad \begin{cases} \text{timelike: } (\Delta s)^2 > 0 \\ \text{spacelike: } (\Delta s)^2 < 0 \end{cases} \quad \text{null: } (\Delta s)^2 = 0$$

proper time (measured by a watch moving along $W$) is $\Delta \tau \equiv \frac{\Delta s}{c}$

proper distance (measured between two events in a ref. frame where they occur simultaneously) is $\Delta L = \sqrt{-(\Delta s)^2}$

For a flat, expanding universe, we have Robertson-Walker metric

$$(ds)^2 = (c\,dt)^2 - (dr)^2 - (r\,d\theta)^2 - (r\sin\theta\,d\phi)^2$$

$$\equiv (c\,dt)^2 - a^2(t)\left[(dx)^2 + (x\,d\theta)^2 + (x\sin\theta\,d\phi)^2\right]$$

where we have defined comoving radius $x$ & scale factor $a$:

$r(t) \equiv a(t)x$ , $r$ is called physical/proper coordinate.

$a(t_0) \equiv 1$ , current day scale factor is 1.

This definition directly gives Hubble-Lemaitre Law, see p2.

**Redshift:** light travels along a null geodesic [i.e., frozen proper time]

$$dx = \frac{c\,dt}{a(t)}$$

consider two photons emitted from comoving distance $x_e$ to $x_0$

$$\int_{t_e}^{t_0} \frac{c\,dt}{a(t)} = \int_{x_e}^{x_0} dx = \int_{t_e+\Delta t_e}^{t_0+\Delta t_0} \frac{c\,dt}{a(t)} = \int_{t_e}^{t_0} - \int_{t_e}^{t_e+\Delta t_e} + \int_{t_0}^{t_0+\Delta t_0}$$

$$\Rightarrow \frac{\Delta t_0}{a(t_0)} - \frac{\Delta t_e}{a(t_e)} = 0 \Rightarrow \frac{1}{a(t_e)} = \frac{\Delta t_0}{\Delta t_e} = \frac{\lambda_0/c}{\lambda_e/c} \equiv (1+z)$$

cosmological redshift is a time dilation effect.

# A6782 F21: Cosmology p2

**Distances:** consider a photon emitted by a comoving object at time $t_e$ & observed by us at $t_{obs}$, what's the [comoving distance] between them?

$$\chi = \int_{t_e}^{t_{obs}} \frac{c\,dt'}{a(t')} = c\int_a^1 \frac{da'}{\dot{a}'\,a'} = c\cdot\int_a^1 \frac{da'}{H(a')a'^2}$$

where we have used Hubble's law: [velocity between two comoving objs]

$$v(t) = \boxed{\frac{dr}{dt} = \frac{\chi\,da}{dt} = \frac{\dot{a}}{a}\cdot r(t)} \equiv H(t)\cdot r(t)$$

$H(t)$: Hubble par.; $r(t)$: proper distance at $t$.

we already have the relation between $a$ & $z$, so we can replace $a$

$$a = \frac{1}{(1+z)} \Rightarrow da = -(1+z)^{-2}\,dz$$

$$\chi = c\cdot\int_z^0 \frac{-(1+z')^{-2}dz'}{H(z')\cdot(1+z')^{-2}} = c\int_0^z \frac{dz'}{H(z')}$$

our comoving distance to the [horizon] is

$$r_{hor} = c\int_0^\infty \frac{dz'}{H(z')} = c\cdot\int_0^{t_{obs}} \frac{dt'}{a(t')}$$

Hubble law distance is comoving distance

$$d_H = r(t_0) \equiv a(t_0)\,\chi \equiv \frac{v(t_0)}{H(t_0)} \left[= \frac{c}{H_0}\ln(1+z)\right]$$

where $(1+z) = e^{v/c}$ for [zero-density model] $[H(a) = H_0/a]$

derivation: $$v = \chi\frac{da}{dt} = c\cdot\int_a^1 \frac{da'}{H(a')a'^2}\cdot\dot{a}(t_0)$$
$$= c\cdot\int_a^1 \frac{da'}{H_0\,a'}\cdot H_0 = c\cdot(0-\ln a) = c\cdot\ln(1+z)$$

Note that the above differs from the relativistic Doppler shift law

$$(1+z) = \sqrt{(1+v/c)/(1-v/c)}$$

and the Hubble law distance is identical to comoving distance, so it depends on cosmology parameters.

## Cosmological Distances :

FLRW Metric : $ds^2 = (c\,dt)^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$

Proper distance (along radial direction) [ $dt=0$, $d\theta = d\phi = 0$ ]

vs. coming dist.

$$s(t_0) \equiv \int_0^s ds' = a(t_0)\int_0^r \frac{dr}{\sqrt{1-kr^2}}$$

$$= a(t_0)\begin{cases} \frac{1}{\sqrt{k}}\sin^{-1}(x\sqrt{k}) & \text{for } k>0 \text{, closed universe} \\ x & \text{for } k=0 \text{, flat} \\ \frac{1}{\sqrt{|k|}}\sinh^{-1}(r\sqrt{|k|}) & \text{for } k<0 \text{, open} \end{cases}$$

$\Rightarrow$ $\boxed{d_p = a(t_0)\cdot x}$ (for flat universe only) where $x$ is comoving distance & $t_0$ is time of measurement

Horizon distance ~~&~~ (comoving distance to ~~an observed event~~ the farthest observable event)

photons move along null geodesic $\Rightarrow ds = 0$

$$\int_0^t \frac{c\,dt}{a(t)} = \int_0^{x_h} \frac{dr}{\sqrt{1-kr^2}}$$

$$\Rightarrow \quad x_h = \begin{cases} \sin\left(c\cdot\int_0^t \frac{dt}{a(t)}\right) & \text{for } k=1 \\ c\cdot\int_0^t \frac{dt}{a(t)} & \text{for } k=0 \\ \sinh\left(c\cdot\int_0^t \frac{dt}{a(t)}\right) & \text{for } k=-1 \end{cases}$$

Today : [CO 29.159]
$x_{h,0} = 14.6\,\text{Gpc}$ $\gg c\cdot t_{\text{Hubble}}$
In the future :
$x_h \to 19.3\,\text{Gpc}$
i.e. it converges to a maximum

Angular diameter distance & Luminosity distance

Photon power loss due to expansion :

$$P_{em} = \frac{h\nu_e}{\delta t_e}, \quad P_{obs} = \frac{h\nu_0}{\delta t_0} = \frac{h\nu_e}{\delta t_e}\cdot\frac{a(t_e)^2}{a(t_0)^2} = P_e\cdot a(t_e)^2$$

$$\Rightarrow F_{obs} = \frac{L_e\cdot a(t_e)^2}{4\pi\, d_p^2} = \frac{L_e}{4\pi\cdot x^2/a(t_e)^2} \equiv \frac{L_e}{4\pi\, d_L^2}$$

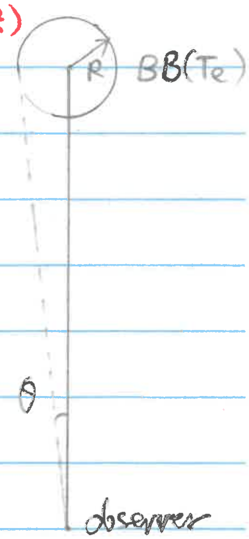
$$\Rightarrow d_L = x/a(t_e) = x\cdot(1+z) = x/a(t_e)$$

$$d_A = d_L/(1+z)^2 = x/(1+z) = x\cdot a(t_e)$$

# A6782 F21: Cosmology p3

distance square law & inverse distance law of expanding universe $\left\{ \begin{array}{l} F = \dfrac{L}{4\pi d_p^2 (1+z)^2} \\ \theta = \dfrac{\ell}{d_p/(1+z)} \end{array} \right.$

## Luminosity & Angular Diameter Distances




$$d_L = \sqrt{\frac{L}{4\pi F}}, \quad d_A = \frac{\ell}{\theta}$$

$$d_L = \chi \cdot (1+z) = c \cdot (1+z) \cdot \int_0^z \frac{dz'}{H(z')}$$

$$d_A = d_L/(1+z)^2 = \chi/(1+z)$$

derivation of $d_A$-$d_L$ relation:

luminosity of BB: $L = 4\pi R^2 \cdot \sigma T_e^4$

surface brightness seen by observer: $I = \frac{\sigma T_{obs}^4}{\pi}$

flux received by observer: $F = \frac{\pi R^2}{d_A^2} \cdot I = \frac{R^2}{d_A^2} \cdot \sigma T_{obs}^4$

definition of $d_L$: $4\pi d_L^2 \cdot F = L = 4\pi R^2 \sigma T_e^4$

$$\Rightarrow 4\pi d_L^2 \cdot \frac{R^2}{d_A^2} \cdot \sigma T_{obs}^4 = 4\pi R^2 \cdot \sigma T_e^4$$

$$\Rightarrow \frac{d_L}{d_A} = \left(\frac{T_{em}}{T_{obs}}\right)^2 = (1+z)^2 = \left[\frac{1}{a(t_e)}\right]^2$$

## Surface brightness ~~dif~~ dimming:

$$I \propto \frac{F}{\theta^2} \propto \frac{L}{d_L^2} \cdot \frac{d_A^2}{\ell^2} = \frac{L}{\ell^2} \cdot (1+z)^{-4}$$

## adiabatic expansion of photon gas

energy density: $u = a \cdot T^4$ (where $a = \frac{4\sigma_{SB}}{c}$), radiative pressure: $P = \frac{1}{3} a T^4$

1st law of TD: $dQ = dE + PdV$, adiabatic: $dQ = 0$

$$\frac{dE}{dt} = -P\frac{dV}{dt} \Rightarrow 4T^3 \cdot V \cdot \frac{dT}{dt} + T^4 \frac{dV}{dt} = -\frac{1}{3} T^4 \cdot \frac{dV}{dt}$$

$$\Rightarrow \frac{1}{T}\frac{dT}{dt} = -\frac{1}{3V}\frac{dV}{dt} = -\frac{1}{a}\frac{da}{dt}, \quad [V = x^3 a^3(t) \text{ is used}]$$

$$\Rightarrow T \propto a^{-1} = (1+z)$$

# Solutions of the Friedmann Equation

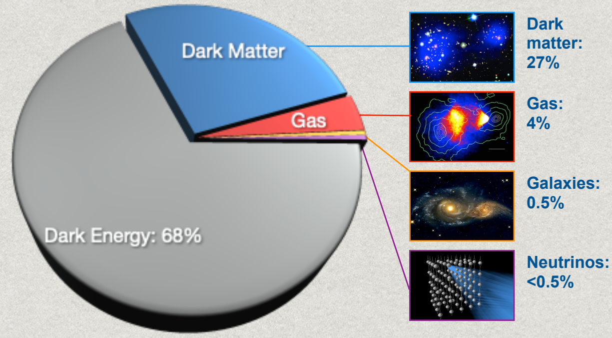
## Composition Today

$$\rho_c(t) = \frac{3H^2(t)}{8\pi G}$$

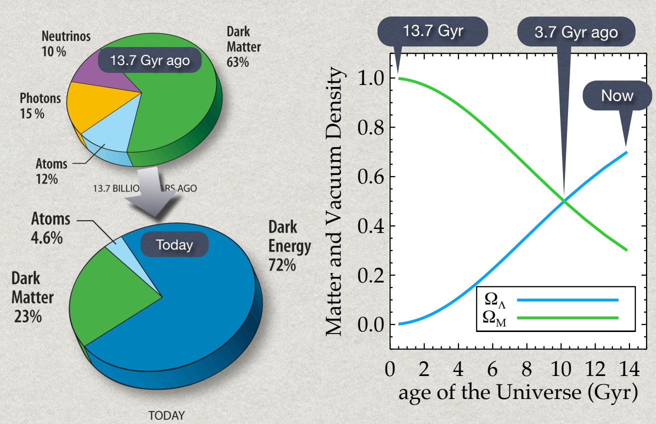
Critical density as a function of time. Value below is present value, based on present value of the Hubble parameter H

$$\rho_{c,0} = \frac{3H_0^2}{8\pi G} = 9.47x10^{-27}\, kg/m^3$$

Planck Collaboration (2013)




## The Composition Varies over Time




## evolution of Omega's

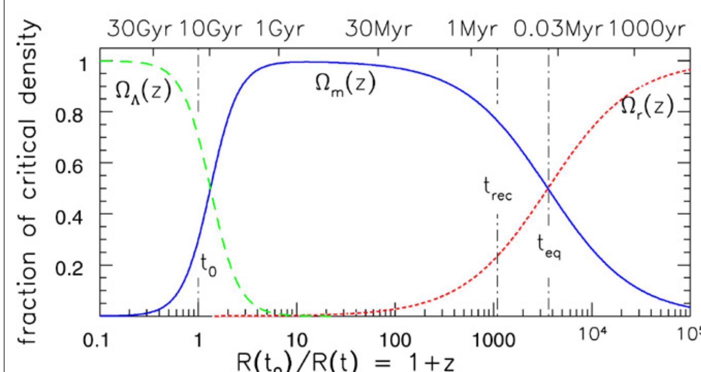
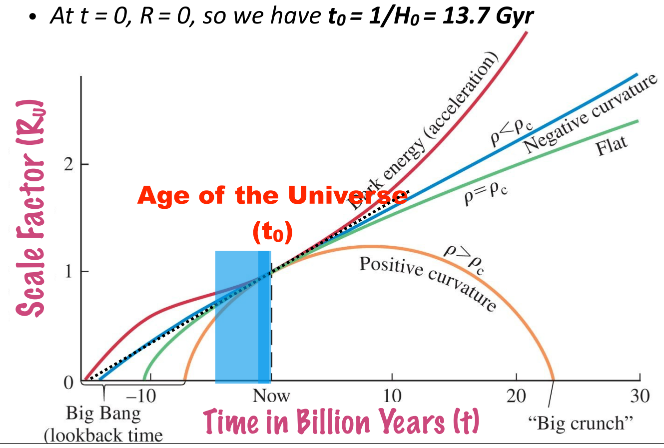



Fig 8.7 'Galaxies in the Universe' Sparke/Gallagher CUP 2007

- for $z \ll 1$, $z \times c = \boldsymbol{v_r = H_0 D} = H_0(t_0\text{-}t)\, c$, so $\boldsymbol{z = H_0(t_0\text{-}t)}$
- for $z \ll 1$, $R = 1/(1+z) \approx 1\text{-}z$, so $\boldsymbol{R = 1\text{-}H_0(t_0\text{-}t)}$
- *At t = 0, R = 0, so we have* $\boldsymbol{t_0 = 1/H_0 = 13.7\ Gyr}$



## Friedmann Eq. (Energy conservation)

imagine an expanding shell with a comoving radius of $x \equiv r/a$

$$E = \frac{1}{2} v^2 - G\frac{M(r)}{r} \quad \text{, where } v = \frac{dr}{dt} = x \cdot \dot{a} \text{ , } M(r) = \frac{4}{3}\pi \rho \cdot (ax)^3$$

$$= \frac{1}{2} H^2 a^2 x^2 - \frac{4}{3}\pi G \rho a^2 x^2$$

$$\equiv -\frac{1}{2} k c^2 x^2 \quad \leftarrow \text{energy per unit mass } E/m \sim c^2$$

$\Rightarrow$

$$\left[\left(\frac{1}{a}\frac{da}{dt}\right)^2 - \frac{8}{3}\pi G \rho\right] a^2 = -kc^2$$

define critical density: $\rho_c \equiv \dfrac{3H^2}{8\pi G}$ $\left[\begin{array}{l} H_0 = 70 \text{ km/s/Mpc} \\ \rho_{c,0} = 10^{-29} \text{ g/cm}^3 \end{array}\right]$

$$\Rightarrow \quad H^2 [1 - \rho/\rho_c) a^2 = -kc^2$$

$\Uparrow$ $\Omega \leftarrow$ density parameter

Full GR version:

$$\left[\left(\frac{1}{a}\frac{da}{dt}\right)^2 - \frac{8}{3}\pi G(\rho_m + \rho_{rel}) - \frac{1}{3}\Lambda c^2\right] a^2 = -kc^2$$

or $$H^2 [1 - (\Omega_m + \Omega_{rel} + \Omega_\Lambda)] a^2 = -kc^2$$

where $\Omega_m = \dfrac{\rho_m}{\rho_c}$ , $\Omega_{rel} = \dfrac{\rho_{rel}}{\rho_c}$ , $\Omega_\Lambda = \dfrac{\Lambda c^2}{8\pi G \rho_c} = \dfrac{\Lambda c^2}{3H^2}$

## Evolution of $\Omega$'s (density parameters)

$$\begin{cases} \dfrac{\rho_m}{\rho_{m,0}} = \dfrac{1}{a^3} \Rightarrow \dfrac{\Omega_m}{\Omega_{m,0}} = \dfrac{\rho_m}{\rho_{m,0}} \cdot \dfrac{\rho_{c,0}}{\rho_c} = \dfrac{\rho_m}{\rho_{m,0}} \dfrac{H_0^2}{H^2} = \dfrac{1}{a^3}\dfrac{H_0^2}{H^2} \\ \dfrac{\rho_r}{\rho_{r,0}} = \dfrac{1}{a^4} \Rightarrow \dfrac{\Omega_r}{\Omega_{r,0}} = \dfrac{1}{a^4}\dfrac{H_0^2}{H^2} \\ \dfrac{\Omega_\Lambda}{\Omega_{\Lambda,0}} = \dfrac{\Lambda H_0^2}{\Lambda H^2} = \dfrac{H_0^2}{H^2} \end{cases}$$

Boundary Condition:

at $t = t_0$, $H_0^2(1-\Omega_0) = -kc^2$, $\Omega_0 = \Omega_{m,0} + \Omega_{r,0} + \Omega_{\Lambda,0}$

$$\Rightarrow H^2a^2[1-(\Omega_m + \Omega_r + \Omega_\Lambda)] = H_0^2(1-\Omega_0)$$

H(a) relation:

$$H^2 - H^2\Omega_m - H^2\Omega_r - H^2\Omega_\Lambda = H_0^2(1-\Omega_0)/a^2$$

$H^2\Omega_m \rightarrow \Omega_{m,0}H_0^2/a^3$, $H^2\Omega_r \rightarrow \Omega_{r,0}H_0^2/a^4$, $H^2\Omega_\Lambda \rightarrow \Omega_{\Lambda,0}H_0^2$

$$\Rightarrow H^2 = H_0^2/a^2\left[(1-\Omega_0) + \Omega_{m,0}/a + \Omega_{r,0}/a^2 + \Omega_{\Lambda,0}a^2\right]$$

so for empty universe ($\Omega_0 = 0$), we have:

$$H = H_0/a$$

for EdS universe ($\Omega_0 = \Omega_{m,0} = 1$, $\Omega_{r,0} = \Omega_{\Lambda,0} = 0$):

$$H = H_0/a^{3/2} \quad \text{Einstein-de Sitter.}$$

$\Omega(a)$ relation:

$$\Omega_m = \frac{\Omega_{m,0}H_0^2}{a^3 \cdot H^2} = \frac{\Omega_{m,0}}{a\left[(1-\Omega_0) + \Omega_{m,0}/a + \Omega_{r,0}/a^2 + \Omega_{\Lambda,0}a^2\right]}$$

for EdS universe, $\Omega_m = \Omega_{m,0}$ at all time.

for $\Lambda$CDM, all $\Omega$'s vary with time in a complex way

$$\Omega_\Lambda = \frac{\Omega_{\Lambda,0}H_0^2}{H^2} = \frac{\Omega_{\Lambda,0}\,a^2}{(1-\Omega_0) + \Omega_{m,0}/a + \Omega_{r,0}/a^2 + \Omega_{\Lambda,0}a^2}$$

## $t(a)$ relation:

consider only $k=0$ case

$$\left(\frac{1}{a}\frac{da}{dt}\right)^2 - \frac{8}{3}\pi G\left(\rho_{m,0}/a^3 + \rho_{r,0}/a^4 + \rho_{\Lambda,0}\right) = 0$$

$$\Rightarrow (dt)^2 = \frac{3a^2(da)^2}{8\pi G(\rho_{m,0}\cdot a + \rho_{r,0} + \rho_{\Lambda,0}a^4)}$$

$$= \frac{1}{H_0^2}\cdot\frac{3H_0^2}{8\pi G}\cdot\frac{a^2(da)^2}{\rho_{m,0}a + \rho_{r,0} + \rho_{\Lambda,0}a^4}$$

$$= t_H^2\,\frac{a^2(da)^2}{\Omega_{m,0}a + \Omega_{r,0} + \Omega_{\Lambda,0}a^4}$$

$$\Rightarrow \quad t = t_H\cdot\int_0^a \frac{a\,da}{\sqrt{\Omega_m a + \Omega_{r,0} + \Omega_{\Lambda,0}a^4}}$$

① EdS universe: $\Omega_0 = \Omega_{m,0} = 1$

$$t = t_H\cdot\int_0^a \sqrt{a}\,da = \frac{2}{3}a^{3/2}t_H \Rightarrow a = \left(\frac{3}{2}\right)^{2/3}\left(\frac{t}{t_H}\right)^{2/3}$$

$\Rightarrow t(a=1) = \frac{2}{3}t_H$ age of the Universe

$\Rightarrow$ lookback time: $t(a=1) - t(a) = \frac{2}{3}t_H(1-a^{3/2})$

② ~~empty~~ radiation only universe: $\Omega_0 = \Omega_{r,0} = 1$

$$t = t_H\cdot\int_0^a a\,da = \frac{1}{2}a^2$$

③ $\Lambda$-only universe: $\Omega_0 = \Omega_{\Lambda,0} = 1$

$$t = t_H\cdot\int_0^a \frac{da}{a} = t_H(\ln a - \ln a_{min})$$

$\Rightarrow$ exponential growth: $a = a_{min}\exp(t/t_H)$