Improved limits on small-scale anisotropy in cosmic microwave background

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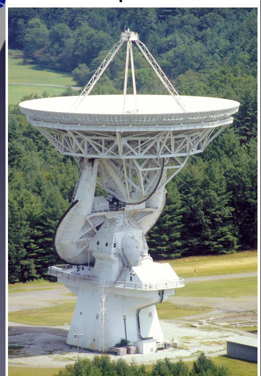
As a remnant of the early Universe, the cosmic microwave background provides unique information on the initial conditions from which matter has evolved to form the structures we see today. All efforts to detect small-scale structure in this radiation have so far been unsuccessful (ref. 1 and refs therein)[1–4]. Nevertheless, upper limits set on possible underlying fluctuations restrict the range of physical models for perturbations of the density in the early Universe. Our search for small-scale anisotropy in the background radiation has now resulted in a lowering of the upper limit on root-mean-square fluctuations ($\Delta T_{r.m.s.}$) observed at an angular scale of ~4 arc min to $\Delta T_{r.m.s.}/T < 2.1\times10^{-5}$ at the 95% confidence level (where $T = 2.7$ K, the temperature of the background radiation). The actual limits deduced from our experiment depend on the model assumed for the unseen fluctuations. Several possibilities are discussed as well as the implications this new measurement has for various cosmological models.

140-ft telescope in Green Bank

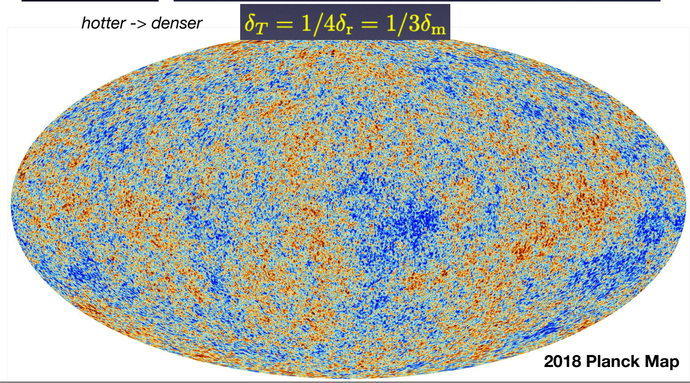


CMB anisotropy shows density fluctuations of 10^{-5} at z_{rec} ~ 1000

isentropic/adiabatic perturbations: entropy per unit mass is preserved:

$$S = \frac{s}{\rho_m} \propto \frac{\rho_r^{3/4}}{\rho_m} \Rightarrow \delta_S = \frac{\partial S}{S} = \frac{1}{S}\left[\frac{\partial S}{\partial \rho_r}\partial\rho_r + \frac{\partial S}{\partial \rho_m}\partial\rho_m\right] = \frac{3}{4}\delta_r - \delta_m$$

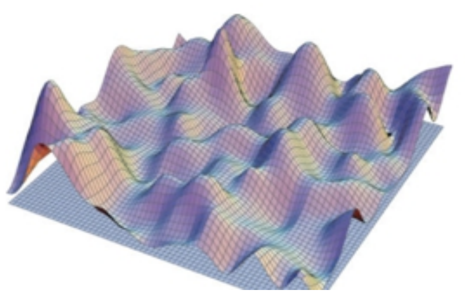
hotter -> denser $\delta_T = 1/4\delta_r = 1/3\delta_m$



2018 Planck Map

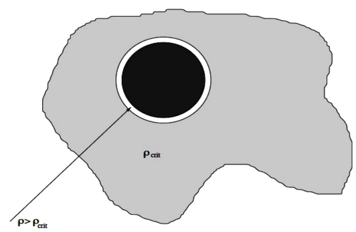
DEFINE DENSITY FLUCTUATION FIELD δ:

δ= (ρ - <ρ>)/ <ρ>



Kauffmann

Top-Hat Spherical Collapse

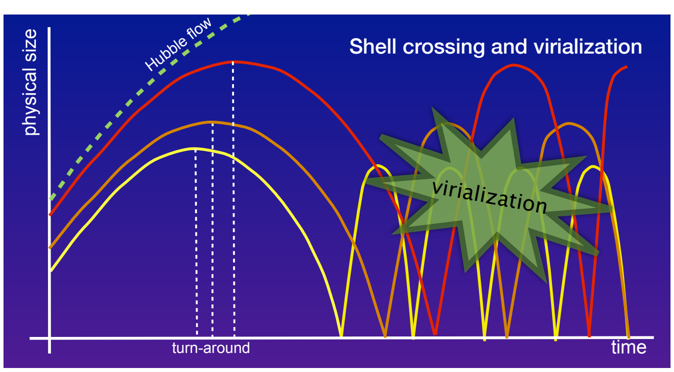


Consider the idealised case of a spherical volume where the density is infinitesimally higher than the cosmic mean.

Our density perturbation will then evolve like a closed universe with $\Omega_m = 1 + \delta$. The scale factor a(t) of such a universe reaches a maximum value a_{max} and then decreases again—in other words, our perturbation will grow to a maximum size $r=r_{max}$ at time $t=t_{max}$ and then collapse.

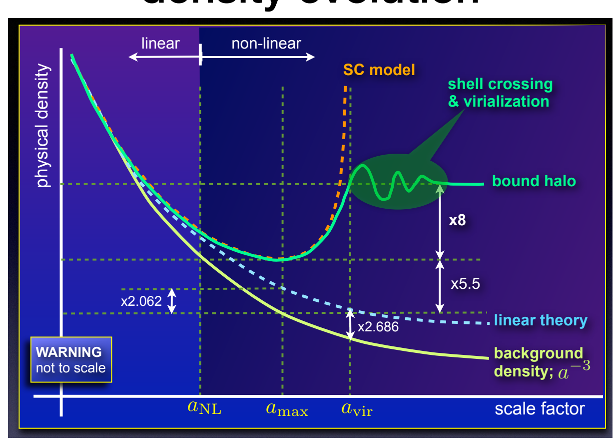
Kauffmann

size evolution



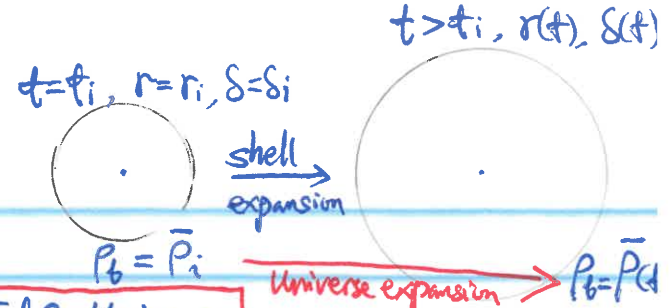
van den Bosch

density evolution



van den Bosch

Formation of DM Halos



Top-Hat Spherical Collapse in $\boxed{\text{EdS Universe}}$.

$$\Omega(t)=\Omega_m(t)=1\,,\quad H=H_0\cdot a^{-3/2}\,,\quad a=\left(\frac{3}{2}\right)^{2/3}\left(\frac{t}{t_H}\right)^{2/3}$$

$$\bar{\rho}(t)=\rho_c=\frac{3H^2}{8\pi G}=\frac{1}{6\pi G t^2}=\frac{3H_0^2}{8\pi G}\cdot a^{-3}\,,\quad t_0-t(a)=\frac{2}{3}t_H(1-a^{3/2})$$

Density contrast: $\quad \delta(t)=\dfrac{\rho(t)-\bar{\rho}(t)}{\bar{\rho}(t)}\quad$ or $\quad 1+\delta=\dfrac{\rho}{\bar{\rho}}$

Mass conservation: $\quad M=\frac{4}{3}\pi r_i^3\,\bar{\rho}_i\,(1+\delta_i)=\frac{4}{3}\pi r^3(t)\,\bar{\rho}(t)\left[1+\delta(t)\right]$

the above eq. shows that in order for δ to evolve, the expansion of the shell must decouple from the cosmic expansion (i.e., Hubble flow) $\Rightarrow\ r(t)\ \cancel{\propto}\ a\propto t^{2/3}$

Energy conservation:

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2-\frac{GM}{r}=E\quad\text{, where } E \text{ is the specific energy of shell}$$

Solution in parametric form: $r(\theta)$ & $t(\theta)$ instead of $r(t)$ or $t(r)$

$$\frac{dr}{dt}=\left(2E+\frac{2GM}{r}\right)^{1/2}\equiv\left[-\left(\frac{A}{B}\right)^2+\frac{2A^3}{B^2}\frac{1}{r}\right]^{1/2}$$

$$=\frac{A}{B}\left(\frac{2A}{r}-1\right)^{1/2}$$ define A: size par, B: time par

$B=A/\sqrt{-2E}$

where we defined two positive constants:

$$A=-\frac{GM}{2E}\quad\&\quad B=\frac{GM}{(-2E)^{3/2}}\,,\quad E<0 \text{ for bounded sys.}$$

$$\Rightarrow dt/B=2\cdot\frac{d(r/2A)}{\sqrt{\frac{2A}{r}-1}}$$ define $r=2A\sin^2\phi$ because $2A$ is the max radius when $KE=0$

$$=\frac{2\,d\sin^2\phi}{\left(\frac{1}{\sin^2\phi}-1\right)^{1/2}}=4\sin^2\phi\,d\phi=(1-\cos\underbrace{2\phi}_{\theta})\,d(\underbrace{2\phi}_{\theta})$$

$$\Rightarrow t=B(\theta-\sin\theta)\quad\&\quad r=A(1-\cos\theta)$$

initial condition: $\theta=0,\ t=0,\ r=0$

The solution implies

shell expands from $r=0$ at $t=0$ $[\theta=0]$

reaches max radius $r_{max} = 2A$ at $t=t_{max}=\pi B$ $[\theta=\pi]$

collapse back to $r=0$ at $t=t_{coll}=2\pi B$ $[\theta=2\pi]$

t_{max} is the turn-around time

t_{coll} is the virialization time

The same solution can be obtained from Friedmann Equation (matter-only)

$$\frac{1}{2}\left(\frac{da}{dt}\right)^2 - \frac{4}{3}\pi G\rho a^2 = -\frac{1}{2}kc^2 = \frac{1}{2}H_i^2(1-\Omega_i)a_i^2$$

here a is equivalent to $r = a \cdot r_0$ where r_0 is comoving radius, and a is scale factor, thus r is physical radius.

rearrange:

$$\frac{1}{2}\left(\frac{da}{dt}\right)^2 - \frac{\frac{4}{3}\pi G\rho_i a_i^3}{a} = \frac{1}{2}H_i^2 a_i^2(1-\Omega_i)$$

$$\Rightarrow \quad GM \equiv \frac{4}{3}\pi G\rho_i a_i^3 \quad , \quad -2E = H_i^2 a_i^2(\Omega_i - 1) > 0$$

solution again is:

$$a = A(1-\cos\theta) \quad \text{where} \quad A = \frac{GM}{-2E} = a_i \frac{\Omega_i}{2(\Omega_i - 1)} \sim \delta_i^{-1}$$

$$t = B(\theta - \sin\theta) \qquad B = \frac{GM}{(-2E)^{3/2}} = \frac{\Omega_i}{2H_i(\Omega_i - 1)^{3/2}} \sim \delta_i^{-\frac{3}{2}}$$

$\boxed{B \propto \delta_i^{-\frac{3}{2}} \Rightarrow \text{larger perturbations collapse earlier}}$

$A \propto \delta_i^{-1} \rightarrow$ and to smaller radius

Density evolution:

mean density of top-hat: $\rho = \frac{3M}{4\pi a^3} = \frac{3M}{4\pi A^3}(1-\cos\theta)^{-3}$

mean density of background (EdS) $\bar{\rho} = \frac{1}{6\pi G t^2} = \frac{1}{6\pi G B^2}(\theta - \sin\theta)^{-2}$

$\Rightarrow$ density contrast

$$1+\delta = \frac{\rho}{\bar{\rho}} = \frac{9}{2}\frac{(\theta-\sin\theta)^2}{(1-\cos\theta)^3} \quad \text{because } A^3/B^2 = GM$$

at turn-around, $\theta = \pi \Rightarrow (1+\delta)_{ta} = \frac{9}{16}\pi^2 = 5.55$

at virialization, $t = 2t_{max}$, $r = \frac{1}{2}r_{max} = A$,

$$(1+\delta)_{vir} = (1+\delta)_{ta} \times \left(\frac{1}{2}\right)^{-3} / 2^{-2} = 18\pi^2 = 178$$

Virialization radius of DM halo:

Virial theorem: $2K_v + \Phi_v = 0 \Rightarrow K_v = -\frac{1}{2}\Phi_v$

Energy conservation: $E_i = K_i + \Phi_i = E_{ta} = 0 + \Phi_{ta}$

$$E_i = K_v + \Phi_v = \frac{1}{2}\Phi_v$$

$$\Rightarrow \Phi_{ta} = \frac{1}{2}\Phi_v \rightarrow -\frac{GM}{r_{ta}} = -\frac{GM}{2r_v}$$

$$\Rightarrow r_v = \frac{1}{2} r_{ta} = A$$

top-hat density increases by $8\times$ from turn-around to virialization.

Expection of linear growth when $\delta \ll 1$

Assume a background flat universe ($k=0$)

$$H^2 = \frac{8\pi G}{3}\bar{\rho}$$

Inside the overdensity ($k>0$)

$$H^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3}(\bar{\rho} + \delta\rho)$$

$\Rightarrow \delta\rho = \frac{3kc^2}{8\pi G a^2}$

$$\Rightarrow \delta = \frac{\delta\rho}{\bar{\rho}} = \frac{3kc^2}{8\pi G(\bar{\rho}a^2)} \propto \begin{cases} a^2 & \text{when radiation-dominates} \\ a & \text{when matter dominates (i.e. EdS)} \end{cases}$$

Further implications:

① larger perturbations collapse earlier:

$$t_{coll} = 2\pi B \propto \delta_i^{-3/2}$$ → larger perturb collapses earlier

② Only $\delta > 10^{-3}$ perturbations at recombination ($z \sim 1000$) would have collapsed by today

$$a_{coll} = \frac{1}{2}a_{ta} = A = a_i \frac{\Omega_i}{\delta_i}, \quad \Omega_i \sim 1$$

for $a_{coll} < a(\text{today}) = 1$, we have $\delta_i > a_i = 10^{-3}$ ← $z = 1000$

Same result can be derived from $t_{coll} < t_H = \frac{2}{3H_0}$, $t_{coll} = 2\pi B$

Virial Theorem $\quad 2\langle K\rangle + \langle U\rangle = 0$

Proof: define $Q = \sum_{i=1}^{N} \vec{p}_i \cdot \vec{r}_i$ (recall that $\vec{L} = \cancel{\vec{p}\times} \; \vec{r}\times\vec{p}$)

$$\frac{dQ}{dt} = \sum_{i=1}^{N}\left(m_i \frac{d\vec{r}_i}{dt}\cdot\frac{d\vec{r}_i}{dt} + m_i \frac{d^2\vec{r}_i}{dt^2}\cdot\vec{r}_i\right)$$
$$= 2K + \underbrace{\sum_{i=1}^{N}\vec{F}_i\cdot\vec{r}_i}_{\boxed{\text{virial of Clausius}}}$$

What is Q?

$Q = \frac{1}{2}\frac{dI}{dt}$

$\frac{dQ}{dt} = \frac{1}{2}\frac{d^2I}{dt^2}$

$$\frac{dQ}{dt} = \frac{d}{dt}\left(\sum_{i=1}^{N} m_i \frac{d\vec{r}_i}{dt}\cdot\vec{r}_i\right) = \frac{d}{dt}\left(\sum_{i=1}^{N}\frac{1}{2}\frac{d}{dt}(m_i\vec{r}_i\cdot\vec{r}_i)\right) = \frac{1}{2}\frac{d^2I}{dt^2}$$

where $I = \sum_{i=1}^{N} m_i r_i^2$ is the $\boxed{\text{moment of inertia}}$

What is $\sum_i \vec{F}_i\cdot\vec{r}_i$

$\sum_i \vec{F}_i\cdot\vec{r}_i = U$ the potential energy.

For gravitational systems $\vec{F}_i = \sum_{j\neq i}\frac{Gm_im_j(\vec{r}_j-\vec{r}_i)}{|\vec{r}_j-\vec{r}_i|^3} = \sum_{j\neq i} F_{ij}$

$$\text{Virial} = \sum_i \vec{F}_i\,\vec{r}_i = \sum_i\left(\sum_{j\neq i}F_{ij}\right)\cdot\frac{1}{2}\left[(\vec{r}_i+\vec{r}_j)+(\vec{r}_i-\vec{r}_j)\right]$$
$$= \frac{1}{2}\sum_i\left(\sum_{j\neq i}\vec{F}_{ij}\right)\vec{r}_i + \frac{1}{2}\sum_i\left(\sum_{j\neq i}\vec{F}_{ij}\right)\vec{r}_j + \frac{1}{2}\sum_i\sum_{j\neq i}\vec{F}_{ij}\,(\vec{r}_i-\vec{r}_j)$$
$$= \frac{1}{2}\sum_i\left[\sum_{j\neq i}\vec{F}_{ij}\cdot\vec{r}_i - \sum_{j\neq i}\vec{F}_{ji}\,\vec{r}_j\right] + \frac{1}{2}\sum_i\sum_{j\neq i}\left(-\frac{Gm_im_j}{|\vec{r}_j-\vec{r}_i|}\right)$$
$$= 0 + \frac{1}{2}\sum_i\sum_{j\neq i}U_{ij}$$
$$= U$$

$$\sum_{i=1}^{3}\sum_{j\neq i}F_{ij}(\vec{r}_i+\vec{r}_j)$$
$$= F_{12}(r_1+r_2)+F_{13}(r_1+r_3)$$
$$+ F_{21}(r_2+r_1)+F_{23}(r_2+r_3)$$
$$+ F_{31}(r_3+r_1)+F_{32}(r_3+r_2)$$

$$\Rightarrow \quad \frac{dQ}{dt} = 2K + U$$

$\frac{dQ}{dt} = 2K + U = 0$ virial theorem

do time integral over a period of τ. $\quad \langle U\rangle = \frac{1}{\tau}\int_0^\tau U\,dt$

$$\frac{Q(\tau)-Q(0)}{\tau} = \langle 2K\rangle + \langle U\rangle = 0 \quad \text{when } \tau\to\infty$$

because $Q = \sum_i \vec{p}_i\cdot\vec{r}_i$ is bounded for a system that reached an equilibrium or steady-state configurations

Application: Virial mass $\quad \sigma^2 = \frac{GM_{vir}}{R_{vir}} \Rightarrow M_{vir} = \frac{\sigma^2 R_{vir}}{G}$

Faber-Jackson relation: $R \propto \frac{L}{\sigma^2}$, $L = 4\pi R^2 B \Rightarrow L \propto L^2/\sigma^4\cdot B$

Non-Linear vs. Linear Growth (in EdS)

What's the density contrast at the time of collapse?

$$t_{coll} = 2\pi B = \frac{\pi}{H_i} \cdot \delta_i^{-\frac{3}{2}}$$

Implicit assumption: $t_{vir} = t_i + t_{coll} \simeq t_{coll}$

$$a(t_{coll}) = \left(\frac{3}{2}\right)^{2/3} \cdot \left(\frac{t_{coll}}{t_H}\right)^{2/3} = \left(\frac{3\pi}{2}\right)^{2/3} \cdot \left(\frac{H_0}{H_i}\right)^{2/3} \cdot \frac{1}{\delta_i}$$

because $H_i = H_0 \cdot a_i^{-3/2}$, we have

$$a(t_{coll}) = \left(\frac{3\pi}{2}\right)^{2/3} \cdot \frac{a_i}{\delta_i}$$

in Linear growth theory, $\delta \propto a$, therefore

$$\delta(t_{coll}) = \delta_i \cdot \frac{a(t_{coll})}{a_i} = \left(\frac{3\pi}{2}\right)^{2/3} = 2.811$$

Note that the final δ is independent of δ_i, just like in non-linear.

A better approximation, starting from the result of non-linear theory:

$$1+\delta = \frac{9}{2} \frac{(\theta - \sin\theta)^2}{(1-\cos\theta)^3}$$

then apply the first 3 orders of Taylor expansion ($\theta \ll 1$)

$$\left.\begin{aligned} \sin\theta &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \\ \cos\theta &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} \end{aligned}\right\} \Rightarrow \begin{aligned} (\theta - \sin\theta)^2 &\simeq \frac{\theta^6}{36}\left(1 - \frac{\theta^2}{10}\right) \\ (1-\cos\theta)^3 &\simeq \frac{\theta^6}{8}\left(1 - \frac{\theta^2}{4}\right) \end{aligned}$$

$$\Rightarrow 1+\delta = \frac{9}{2} \cdot \frac{8}{36} \frac{(1-\theta^2/10)}{(1-\theta^2/4)} \simeq \left(1 - \frac{\theta^2}{10}\right)\left(1 + \theta^2/4\right) \simeq 1 + \frac{3\theta^2}{20}$$

$$\Rightarrow \delta \simeq 3\theta^2/20$$

on the other hand, $t = B(\theta - \sin\theta) \simeq B\theta^3/6$, & $t_{max} = \pi B$

$$\Rightarrow \theta = \left(\frac{6t}{B}\right)^{1/3} = \left(\frac{6\pi t}{t_{max}}\right)^{1/3}$$

therefore,

$$\delta(t) = \frac{3}{20}(6\pi)^{2/3}\left(\frac{t}{t_{max}}\right)^{2/3} = \begin{cases} 1.062 & @ \ t_{max} = t_{ta} \\ 1.686 & @ \ t_{coll} = 2t_{max} \end{cases}$$

Virial radius, virial mass, circular velocity

r_Δ is the radius of a sphere containing an overdensity with $\rho = \Delta_c \cdot \Omega_m \rho_{crit}$

$$\langle \rho(r_\Delta) \rangle = \Delta_c \cdot \Omega_m \cdot \rho_{crit}, \text{ where } \Delta_c = (1+\delta)_{vir}$$

where Δ_c, Ω_m, & ρ_{crit} are all functions of z, so r_Δ is calculated at the epoch of virialization.

In EdS, $\Delta_c = 18\pi^2 = 178$ at all z, and $\Omega_m(z) = 1.0$

In ΛCDM, $\Delta_c(z) = 18\pi^2 + 82y - 39y^2$, where $y = \Omega_m(z) - 1$

$$\Omega_m(z) = \frac{\Omega_{m,0}(1+z)^3}{\Omega_{m,0}(1+z)^3 + \Omega_{\Lambda,0}}$$

$\Delta_c = 100$ at $z=0$ & $\Delta_c = 178$ at high z ($z>4$)

Define virial mass $\quad M_\Delta = \frac{4\pi}{3} \Delta_c \cdot \rho_{crit} \cdot \Omega_m \cdot r_\Delta^3$

Define circular velocity $\quad V_\Delta = \left(\frac{GM_\Delta}{r_\Delta}\right)^{1/2}$ (recall $2K + U = 0$)

We have $\quad r_\Delta = \left[\frac{2GM_\Delta}{\Delta_c \Omega_m H^2(z)}\right]^{1/3}$

$$V_\Delta = (GM_\Delta H)^{1/3} \cdot \left(\frac{\Delta_c \Omega_m}{2}\right)^{1/6}$$

Again, the above two parameters should be calculated at $z_{virialization}$.

Once virialized & without mergers, r_Δ & V_Δ will remain constant.

For simplicity, often in the literature, $\Delta_c = 200$ is assumed at all z, thus calculated virial radius & mass are indicated by r_{200} & M_{200}

From the solution of spherical collapse, we also have (for EdS):

$$M_{vir} = \frac{4}{3}\pi r_{vir}^3 \cdot (1+\delta)_{vir} \rho_{crit} = \frac{\Delta_c}{2} \frac{H^2}{G} \cdot \left(\frac{a_i \Omega_i}{\delta_i}\right)^3 \cdot r_0^3 \propto (r_0/\delta_i)^3$$