Halo Formation: Top-Hat Spherical Collapse Model

ASTR:6782 Hai Fu Cosmological Framework: A Simple, Naive Picture

Fundamental Assumption: The Cosmological Principle

- Although galaxies tend to clump, on the largest cosmic scales, the Universe is both **homogeneous** and **isotropic**
 - Homogeneous: there is no preferred location in the Universe
 - Isotropic: there is no preferred direction in the Universe



Robertson-Walker Metric: Differential Space-Time Distance

• In **General Relativity**, a **metric** is a function which measures *differential space-time distance* between two events:

$$(ds)^2 = (c \cdot dt)^2 - (dl)^2$$

• The **Robertson-Walker metric** is the metric that describes the geometry of a **homogeneous**, **isotropic**, **expanding** universe. The metric in *spherical coordinate system* is:

$$(ds)^{2} = (c \cdot dt)^{2} - R_{U}^{2}(t) \left[\left(\frac{dx}{\sqrt{1 - kx^{2}}} \right)^{2} + (xd\theta)^{2} + (x\sin\theta d\phi)^{2} \right]$$

where

R_U is the **scale factor**, defined to be 1 at present day, and <1 in the past **x** is the **comoving** radial distance, $x \equiv r(t)/R_U(t)$, **k** is the **comoving curvature**, $k \equiv \frac{\kappa}{R^2}$, where $\kappa = +1,0, -1$ for positive, flat, and negative curvatures

R is the **comoving** radius of the curvature.

Friedmann's Equations Derived from GR

Einstein's Field Equations derived from the principle of least action (see Landau & Lifshitz, The Classical Theory of Fields, 1975 Edition)

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R - \Lambda g_{\alpha\beta} = \frac{8\pi G}{c^4}\mathcal{T}_{\alpha\beta}$$

RW Metric -> Friedmann's Equations:

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi}{3}G\rho + \frac{\Lambda c^2}{3}$$
$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = -\frac{8\pi}{c^2}Gp + \Lambda c^2$$

Alternative Form of FE1:

$$H \equiv \frac{\dot{a}}{a}, \ \rho_c \equiv \frac{3H^2}{8\pi G}, \ \Omega_i \equiv \frac{\rho_i}{\rho_c}, \ \rho_\Lambda \equiv \frac{\Lambda c^2}{8\pi G}$$
$$a^2 H^2 \left[1 - (\Omega_m + \Omega_\gamma + \Omega_\Lambda)\right] = H_0^2 \left[1 - (\Omega_{m,0} + \Omega_{\gamma,0} + \Omega_{\Lambda,0})\right]$$

The Distribution of Galaxies is Neither Homogenous nor Isotropic

The Milky Way Galaxy

light travel time from Earth:



Simulation vs. Observations





EVOLUTION OF THE UNIVERSE (ILLUSTRATION)

The Earliest Deviations from a Homogeneous Isotropic Universe

NASA rewrites history of CMB discovery



Improved limits on small-scale anisotropy in cosmic microwave background

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As a remnant of the early Universe, the cosmic microwave background provides unique information on the initial conditions from which matter has evolved to form the structures we see today. All efforts to detect small-scale structure in this radiation have so far been unsuccessful (ref. 1 and refs therein)¹⁻⁴. Nevertheless, upper limits set on possible underlying fluctuations restrict the range of physical models for perturbations of the density in the early Universe. Our search for small-scale anisotropy in the background radiation has now resulted in a lowering of the upper limit on root-mean-square fluctuations ($\Delta T_{r.m.s}$) observed at an angular scale of ~4 arc min to $\Delta T_{r.m.s}/T < 2.1 \times 10^{-5}$ at the 95% confidence level (where T = 2.7 K, the temperature of the background radiation). The actual limits deduced from our experiment depend on the model assumed for the unseen fluctuations. Several possibilities are discussed as well as the implications this new measurement has for various cosmological models.

140-ft telescope in Green Bank



The All-sky Temperature Map of the CMB in Mollweide (equal-area) projection



After subtracting the Milky Way, there is a strong dipole signal in the CMB (mean *T* = 2.7K)

Dipole Maximum Direction: RA = 167.942 ± 0.007 , Dec = -6.944 ± 0.007 (J2000).

Dipole Maximum Amplitude: 3.362 mK



What is the relative velocity between the Solar System and the CMB rest frame?

Subtraction of the dipole reveals smaller scale fluctuations in the CMB (the anisotropies)

G X U V I R

Improved angular resolutions over three generations of satellites

CMB anisotropy shows density fluctuations of 10⁻⁵ at z ~ 1000

Evolution of Inhomogeneity

dark matter simulation of a comoving volume that is 40 Mpc across

Cosmological Framework: EdS Universe

Fig 8.7 'Galaxies in the Universe' Sparke/Gallagher CUP 2007

Einstein-de Sitter Universe is a Good Approximation for the Bulk of the Universe's History — the framework of our gravitational collapse model

• Given the dimensionless Hubble parameter from FE1:

$$E(a) = \frac{H}{H_0} = \sqrt{(1 - \Omega_0)/a^2 + \Omega_{m,0}/a^3 + \Omega_{\gamma,0}/a^4 + \Omega_{\Lambda,0}}$$

• and the rearranged time-scale factor relation:

$$\frac{t}{t_H} = \int_0^{1/(1+z)} \frac{da}{E(a)a} = \int_z^\infty \frac{dz'}{(1+z')E(z')}$$

A matter-only, flat universe is known as the Einstein-de Sitter universe. It has the following density parameters:
Ω₀ = Ω_{m,0} = 1, Ω_{γ,0} = Ω_{Λ,0} = 0
which lead to the following analytical solution:

$$\Rightarrow H = H_0 a^{-3/2}$$
$$\Rightarrow t = \frac{2}{3} t_H a^{3/2}$$
$$\Rightarrow \rho_c = \frac{3H^2}{8\pi G} = \rho_{c,0} a^{-3} = \frac{1}{6\pi G t^2}$$

Random Density Fluctuation Field:

$$\delta_{\rho} = \frac{d\rho}{\rho} = d\ln\rho$$

Top-hat spherical collapse

model: consider an idealized spherical volume that happens to have higher than the cosmic mean density

The density perturbation evolves like a separate universe with a slightly different Ω_m

Graphs from Kauffmann

Orange curve shows the expansion history of a super-critical universe

cording to the SC model, $\delta(t_{coll}) = \infty$, which would result in the formation of the base of the second structure of the se

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dividual oscillating shells interact gravitationally, exchanging energy (virial process, to be described in more detail below, results in a virialized dark m

Parameterized Solution of Top-Hat Spherical Collapse

- initial scale factor (inside=outside): $a_i = 1/z_{\text{recombination}} \approx 10^{-3}$
- radius of Top Hat = inside scale factor x comoving radius $\frac{a(\theta)}{a_i} = \frac{1 + \delta_i}{\delta_i} (1 + \cos \theta) = A(1 + \cos \theta)$
- time

$$t(\theta) = \frac{1 + \delta_i}{2H_i \delta_i^{3/2}} (\theta - \sin \theta) = B(\theta - \sin \theta)$$

since $H = H_0 a^{-3/2}$ in **EdS** Universe, $H_i = H_0 a_i^{-3/2}$, we obtain:
$$\frac{t(\theta)}{1/H_0} = \frac{1 + \delta_i}{2a_i^{-3/2} \delta_i^{3/2}} (\theta - \sin \theta)$$

density contrast:

$$1 + \delta(\theta) = \frac{\rho}{\rho_c} = \frac{9}{2} \frac{(\theta - \sin \theta)^2}{(1 - \cos \theta)^3}$$

at turn-around $(\theta = \pi)$: $1 + \delta_{ta} = 9\pi^2/16$
at virialization: $a_v = a_{ta}/2$, $t_v = 2t_{ta} \Rightarrow 1 + \delta_v = 2^5(1 + \delta_{ta}) = 18\pi^2$

Density Evolution of the Overdensity

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The linearly astronalated density field collapses u

The Linear Theory: Simplifying Solutions by Taylor Expansion

• starting from the **density contrast** = 1 + overdensity:

$$1 + \delta(\theta) = \frac{\rho}{\rho_c} = \frac{9}{2} \frac{(\theta - \sin \theta)^2}{(1 - \cos \theta)^3}$$

• when $\theta \ll 1$, we can use **Taylor expansions** to show:

$$1 + \delta(\theta) \approx 1 + \frac{3}{20}\theta^2$$

• when $\theta \ll 1$, we can also express θ as a function of time

$$t = B(\theta - \sin \theta) \approx B\theta^3/6$$
, so $\theta = \left(\frac{6t}{B}\right)^{1/3} = \left(\frac{6\pi t}{t_{ta}}\right)^{1/3}$ where $t_{ta} = \pi B$

• combining the results, we have **overdensity** as function of time:

$$\delta(t) \approx \frac{3}{20} \left(\frac{6\pi t}{t_{ta}}\right)^{2/3}$$

which equals 1.062 at turnaround ($t = t_{ta}$) and 1.686 at virialization ($t = 2t_{ta}$)

• EdS: the outside scale factor is

$$a(t) = \left(\frac{3t}{2t_H}\right)^{2/3}$$
, so that $\delta \propto a = 1/(1+z)$ (Linear growth of overdensity)