This process, to be described in more detail below, results in a virialized dark matter halo of a black hole. However, in reality, the collapse is never perfectly spherical. According to the SC model, the density perturbations obey approximately $\delta = (\rho - \langle \rho \rangle) / \langle \rho \rangle$, which would result in the formation of a virialized dark matter halo after shell crossing and virialization. The physical density $\rho \equiv \rho_m$ of a mass $m$ is not conserved but the entropy $S \equiv S_m$ per unit mass is preserved: $\rho(a)/\rho_0 = a^3 - \delta_m(a)/\delta_m(0)$, and thus $\rho(a)/\rho_0 = a^3 - \delta_m(a)/\delta_m(0)$, and thus $\rho(a)/\rho_0 = a^3 - \delta_m(a)/\delta_m(0)$.

Our density perturbation will then evolve like a closed universe with $\Omega_m = 1 + \delta$. The scale factor $a(t)$ of such a universe reaches a maximum value $a_{\text{max}}$ and then decreases again—in other words, our perturbation will grow to a maximum size $r = a_{\text{max}}$ at time $t = t_{\text{max}}$ and then collapse.

Consider the idealised case of a spherical volume where the density is infinitesimally higher than the cosmic mean. The linearly extrapolated density field collapses when $\delta = 1.686 \times 10^{-2}$. For $\delta > 1$, the density field is no longer a spherical collapse, and then decreases again—in other words, our perturbation will grow to a maximum size $r = a_{\text{max}}$ at time $t = t_{\text{max}}$ and then collapse.
Formation of DM Halos

Top-Hat Spherical Collapse in EoS Universe

\[ \Omega(t) = 5 \ln(t) = 1, \quad H = H_0 \cdot a^{-\frac{1}{2}}, \quad a = \left( \frac{3}{\bar{c}^2} \right) \left( \frac{t}{t_0} \right)^{\frac{1}{2}} \]

\[ \bar{\rho}(t) = \rho_c = \frac{3H_0^2}{8\pi G} = \frac{1}{6\pi G a^2 t^2} = \frac{3H_0^2}{8\pi G} \cdot a^{-3}, \quad t \approx t_0 \approx \frac{2}{3} t_0 (1 - a^{-3}) \]

Density contrast:

\[ \delta(t) = \frac{\rho(t) - \bar{\rho}(t)}{\bar{\rho}(t)} \text{ or } 1 + \delta = \frac{\rho}{\bar{\rho}} \]

Mass conservation:

\[ M = \frac{4}{3} \pi r_1^3 \bar{\rho}(1 + \delta) = \frac{4}{3} \pi r_1^3 (1 + \delta) \]

the above eq. shows that in order for \( \delta \) to evolve, the expansion of the shell must decouple from the cosmic expansion (i.e. Hubble flow) \( \Rightarrow r(t) \propto a \propto t^{\frac{1}{3}} \)

Energy conservation:

\[ \frac{1}{2} \left( \frac{dr}{dt} \right)^2 - \frac{GM}{r} = E \quad \text{where } E \text{ is the specific energy of shell} \]

Solution in parametric form: \( r(\theta) \& t(\theta) \) instead of \( r(t) \) or \( t(r) \)

\[ \frac{dr}{dt} = \left( 2E + \frac{2GM}{r} \right)^{\frac{1}{2}} = \left[ -\left( \frac{A}{B} \right)^2 + \frac{2A^3 \sin^2 \phi}{B^2 r} \right]^{\frac{1}{2}} \]

\[ = \frac{A}{B} \left( \frac{2A}{r} - 1 \right)^{\frac{1}{2}} \quad \text{define } A: \text{size par}, \ B: \text{time par} \]

where we defined two positive constants:

\[ A = -\frac{GM}{2E} \quad \& \quad B = \frac{GM}{(-2E)^{\frac{3}{2}}} \quad \text{E<0 for bounded sys.} \]

\[ \Rightarrow \frac{dt}{B} = 2 \sqrt{\frac{2A}{r} - 1} \quad \text{define } r = 2A \sin^2 \phi \text{ because } 2A \text{ is the max radius when KE=0} \]

\[ = \frac{d \sin^2 \phi}{(\frac{1}{\sin^2 \phi} - 1)^{\frac{1}{2}}} = 4 \sin^2 \phi \sin \phi = (1 - \cos 2\phi) d(2\phi) \]

\[ \Rightarrow t = B(\theta - \sin \theta) \quad \& \quad r = A(1 - \cos \theta) \]

initial condition: \( \theta = 0, \ t = 0, \ r = 0 \)
The solution implies:

- Shell expands from $r = 0$ at $t = 0$ (at $\theta = 0$)
- Reaches max radius $R_{\text{max}}$ at $t = t_{\text{max}} = \frac{2A}{\pi B}$ (at $\theta = \pi$)
- Collapses back to $r = 0$ at $t = t_{\text{coll}} = 2\pi B$ (at $\theta = 2\pi$)
- $t_{\text{max}}$ is the turnaround time
- $t_{\text{coll}}$ is the visualization time

The same solution can be obtained from Friedmann Equation (matter-only):

\[
\frac{1}{a} \left( \frac{da}{dt} \right)^2 = \frac{4}{3} \pi G \rho a^2 = -\frac{1}{2} k c^2 = \frac{1}{2} H_i^2 (1 - S_i) a_i^2
\]

Rearrange:

\[
\frac{1}{2} \left( \frac{da}{dt} \right)^2 - \frac{4\pi G\rho_i a_i^3}{a} = \frac{1}{2} H_i^2 a_i (1 - S_i)
\]

\[\Rightarrow GM = \frac{4\pi G\rho_i a_i^3}{a}, \quad -2E = H_i^2 a_i (S_i - 1) > 0\]

Solution again is:

\[a = A (1 - \cos \theta), \quad \text{where} \quad A = \frac{GM}{-2E} = a_i \frac{S_i}{2(S_i - 1)} \sim S_i^{-1}\]

\[t = B (\theta - \sin \theta)\]

\[B \propto S_i \Rightarrow \text{larger perturbations collapse earlier}\]

\[B = \frac{GM}{-2E} a_i = \frac{S_i}{2H_i (S_i - 1)} \sim S_i^{-\frac{3}{2}}\]

Density evolution:

- mean density of shell: \( \rho = \frac{3M}{4\pi a^3} = \frac{3M}{4\pi A^3} (1 - \cos \theta)^{-3} \)
- mean density of background (Fierz): \( \bar{\rho} = \frac{1}{6\pi G} a^2 = \frac{1}{6\pi G} B^2 (\theta - \sin \theta)^2 \)

\[\Rightarrow \text{density contrast} \quad (1 + \delta) = \frac{\rho}{\bar{\rho}} = \frac{9 (\theta - \sin \theta)^2}{2 (1 - \cos \theta)^3} \text{ because } A^3/B^2 = GM\]

At turnaround, \( \theta = \pi \Rightarrow (1 + \delta)_{\text{ta}} = \frac{9}{16} \pi^2 = 5.55 \)

At visualization, \( t = 2t_{\text{max}}, \quad r = \frac{1}{2} r_{\text{max}} = A, \)

\[(1 + \delta)_{\text{vir}} = (1 + \delta)_{\text{ta}} \times (\frac{1}{2})^{\frac{3}{2}}/2^{-2} = 18\pi^2 = 178\]
Virialization radius of DM halo:

Virial theorem: \( 2K_v + \Phi_v = 0 \Rightarrow K_v = -\frac{1}{2} \Phi_v \)

Energy conservation: \( E_i = K_i + \Phi_i = E_{ta} = 0 + \Phi_{ta} \)
\[ E_i = K_v + \Phi_v = \frac{1}{2} \Phi_v \]

\[ \Rightarrow \Phi_{ta} = \frac{1}{2} \Phi_v \rightarrow -\frac{GM}{R_{ta}} = -\frac{GM}{2R_v} \]

\[ \Rightarrow R_v = \frac{1}{2} R_{ta} = A \]

top-hat density increases by \( 8 \times \) from turn-around to virialization.

Expectation of linear growth when \( S \ll 1 \)

Assume a background flat universe (\( k = 0 \))
\[ H^2 = \frac{8\pi G \bar{\rho}}{3} \]

Inside the overdensity (\( k > 0 \))
\[ H^2 + \frac{k c^2}{a^2} = \frac{8\pi G}{3} (\bar{\rho} + \delta \rho) \]

\[ \Rightarrow S = \frac{\delta \rho}{\bar{\rho}} = \frac{3 kc^2}{8\pi G (\bar{\rho} a^2)} \propto \begin{cases} a^2 & \text{when radiation dominates} \\ a & \text{when matter dominates (i.e. Eds)} \end{cases} \]

Further implications:

1. Larger perturbations collapse earlier:
\[ t_{\text{coll}} = 2\pi B \propto S_i^{-3/2} \Rightarrow \text{larger perturb collapese earlier} \]

2. Only \( S > 10^{-3} \) perturbations would have collapsed by today
   of recombination (\( z \sim 10^3 \))
\[ a_{\text{coll}} = \frac{1}{2} a_{\text{ta}} = A = A_i \frac{S_i}{S_i}, \quad S_i \ll 1 \]

For \( a_{\text{coll}} < a(\text{today}) = 1 \), we have \( S_i > a_i = 10^{-3} \rightarrow z \sim 1000 \)

Same result can be derived from \( t_{\text{coll}} < t_H = \frac{2}{\dot{a}_H}, \quad t_{\text{coll}} = 2\pi B \)
Virial Theorem: \( 2\langle k \rangle + \langle u \rangle = 0 \)

Proof: Define \( Q = \sum_{i=1}^{N} \vec{p}_i \cdot \vec{r}_i \) (recall that \( \vec{L} = \sum_{i=1}^{N} \vec{r}_i \times \vec{p}_i \)).

\[
\frac{dQ}{dt} = \frac{d}{dt} \left( \sum_{i=1}^{N} \left( m_i \frac{d\vec{r}_i}{dt} \cdot \frac{d\vec{r}_i}{dt} + m_i \frac{d^2\vec{r}_i}{dt^2} \cdot \vec{r}_i \right) \right) = 2\dot{K} + \sum_{i=1}^{N} \vec{F}_i \cdot \vec{r}_i
\]

\[
\frac{dQ}{dt} = \frac{d}{dt} \left( \sum_{i=1}^{N} m_i \frac{d\vec{r}_i}{dt} \cdot \vec{r}_i \right) = \frac{d}{dt} \left( \sum_{i=1}^{N} \frac{1}{2} \dot{r}_i \right) = \frac{1}{2} \frac{d^2I}{dt^2}
\]

where \( I = \sum_{i=1}^{N} m_i \dot{r}_i^2 \) is the moment of inertia.

For gravitational systems, \( \vec{F}_i = \sum_{j \neq i} G m_i m_j \frac{(\vec{r}_{ij} \cdot \vec{r}_i)}{|\vec{r}_{ij} - \vec{r}_i|^3} = \sum_{j \neq i} F_{ij} \)

Virial theorem: \( \sum_{i} \vec{F}_i \cdot \vec{r}_i = \sum_{j \neq i} \left( \sum_{i} \sum_{j \neq i} F_{ij} \right) \left( \frac{1}{2} |\vec{r}_i + \vec{r}_j|^2 + (\vec{r}_i - \vec{r}_j) \right) \)

\[
= \frac{1}{2} \sum_{i} \sum_{j \neq i} \left( \sum_{i} \sum_{j \neq i} F_{ij} \right) \dot{r}_i^2 + \frac{1}{2} \sum_{i} \sum_{j \neq i} \left( \sum_{i} \sum_{j \neq i} F_{ij} \right) \dot{r}_j^2 + \frac{1}{2} \sum_{i} \sum_{j \neq i} \sum_{i} \sum_{j \neq i} F_{ij} \left( \dot{r}_i \cdot \dot{r}_j \right)
\]

\[
= \frac{1}{2} \sum_{i} \sum_{j \neq i} \sum_{i} \sum_{j \neq i} F_{ij} \left( \dot{r}_i \cdot \dot{r}_j \right) + \frac{1}{2} \sum_{i} \sum_{j \neq i} \sum_{i} \sum_{j \neq i} \left( \frac{G m_i m_j}{|\vec{r}_{ij}|} \right) \left( \vec{r}_{ij} \cdot \vec{r}_i \right)
\]

\[
= 0 + \frac{1}{2} \sum_{i} \sum_{j \neq i} \vec{U}_{ij} = U
\]

\[
\Rightarrow \quad \frac{dQ}{dt} = 2K + U
\]

Therefore, the integral over a period of \( T \)

\[
\langle U \rangle = \frac{1}{T} \int_0^T U \, dt
\]

Because \( Q = \sum_{i} \vec{F}_i \cdot \vec{r}_i \) is bounded for a system that reached an equilibrium or steady-state configuration.

Application: Virial mass \( \sigma^2 = \frac{G M_{\text{vir}}}{R_{\text{vir}}} \Rightarrow M_{\text{vir}} = \sigma^2 R_{\text{vir}} \).

Faber-Jackson relation: \( R \propto \frac{L}{M}, L = 4\pi R^2 B \Rightarrow L \propto L^2 / \sigma^4 B \).
Non-Linear vs. Linear Growth (in EdS)

What is the density contrast at the time of collapse?

\[ t_{\text{coll}} = 2\pi B = \frac{\pi}{H_i} \cdot \delta; \]

Implicit assumption:
\[ t_{\text{vir}} = t_i + t_{\text{coll}} \approx t_{\text{coll}} \]

\[ a(t_{\text{coll}}) = \left( \frac{3}{2} \right)^{2/3} \left( \frac{t_{\text{coll}}}{t_i} \right)^{2/3} = \left( \frac{3\pi}{2} \right)^{2/3} \frac{H_0}{H_i} \frac{1}{\delta}; \]

because \( H_i = H_0 \cdot a_i^{-\frac{1}{2}} \), we have

\[ a(t_{\text{coll}}) = \left( \frac{3\pi}{2} \right)^{2/3} \frac{a_i}{\delta}; \]

in linear growth theory, \( \delta \propto a \), therefore

\[ \delta(t_{\text{coll}}) = \delta_i \cdot \frac{a(t_{\text{coll}})}{a_i} = \left( \frac{3\pi}{2} \right)^{2/3} = 2.811 \]

Note that the final \( \delta \) is independent of \( \delta_i \), just like in non-linear.

A better approximation; starting from the result of non-linear theory:

\[ 1 + \delta = \frac{9}{2} \left( \frac{\Theta - \sin \Theta}{1 - \cos \Theta} \right)^2 \]

then apply the first 3 orders of Taylor expansion (\( \Theta << 1 \))

\[ \sin \Theta = \Theta - \frac{\Theta^3}{3!} + \frac{\Theta^5}{5!} \]

\[ \cos \Theta = 1 - \frac{\Theta^2}{2!} + \frac{\Theta^4}{4!} \]

\[ \Rightarrow 1 + \delta = \frac{9}{2} \cdot \frac{8}{36} \left( 1 - \frac{\Theta^2}{10} \right) \approx \left( 1 - \frac{\Theta^2}{10} \right) \left( 1 + \frac{\Theta^2}{4} \right) \approx 1 + \frac{3\Theta^2}{20} \]

\[ \Rightarrow \delta \approx 3\Theta^2/20 \]

On the other hand, \( t = B(\Theta - \sin \Theta) = B \Theta^{3/6} \), \& \( t_{\text{max}} = \pi B \)

\[ \Rightarrow \Theta = \left( \frac{6t}{B} \right)^{1/3} = \left( \frac{6\pi t}{t_{\text{max}}} \right)^{1/3} \]

Therefore,

\[ \delta(t) = \frac{3}{20} \left( 6\pi \right)^{2/3} \left( \frac{t}{t_{\text{max}}} \right)^{4/3} = \begin{cases} 1.062 & @ t_{\text{max}} = t_{\text{vir}} \\ 1.686 & @ t_{\text{coll}} = 2t_{\text{max}} \end{cases} \]
Virial radius, virial mass, circular velocity

\( R_\Delta \) is the radius of a sphere containing an overdensity with \( P = \Delta_c \cdot S_m \cdot P_{\text{crit}} \)

\[ \langle P(R_\Delta) \rangle = \Delta_c \cdot S_m \cdot P_{\text{crit}}, \text{ where } \Delta_c = (1 + 8) \nu \]

where \( \Delta_c \), \( S_m \), & \( P_{\text{crit}} \) are all functions of \( z \), so \( R_\Delta \) is calculated at the epoch of virialization.

In EDS, \( \Delta_c = 18 \pi^2 = 178 \) at all \( z \), and \( S_m(z) = 1.0 \)

In \( \Lambda \)CDM, \( \Delta_c(z) = 18 \pi^2 + 82 y - 39 y^2 \), where \( y = S_m(z) - 1 \)

\[ S_m(z) = \frac{S_m(0) (1 + z)^3}{S_m(0)(1 + z)^3 + 2} \]

\( \Delta_c = 100 \text{ at } z = 0 \) & \( \Delta_c = 178 \text{ at high } z (z > 4) \)

Define virial mass \( M_\Delta = \frac{4\pi}{3} \Delta_c \cdot P_{\text{crit}} \cdot S_m \cdot R_\Delta^3 \)

Define circular velocity \( V_\Delta = \left( \frac{G M_\Delta}{R_\Delta} \right)^{1/2} \) (recall \( 2K + U = 0 \))

We have \( R_\Delta = \left[ \frac{2GM_\Delta}{\Delta_c S_m H^2(z)} \right]^{1/3} \)

\[ V_\Delta = (G M_\Delta H)^{1/3} \cdot \left( \frac{\Delta_c S_m}{2} \right)^{1/6} \]

Again, the above two parameters should be calculated at virialization.

Once virialized & without mergers, \( R_\Delta \) & \( V_\Delta \) will remain constant.

For simplicity, often in the literature, \( \Delta_c = 200 \) is assumed at all \( z \), thus calculated virial radius & mass are indicated by \( R_{200} \) & \( M_{200} \)

From the solution of spherical collapse, we also have (for EDS):

\[ M_{\text{vir}} = \frac{4\pi}{3} R_{\text{vir}}^3 (1 + 8) \nu \text{ir} P_{\text{crit}} = \Delta_c \frac{H^2}{2} \left( \frac{\Omega_m}{\Omega_i} \right)^{1/3} R_{\text{vir}}^3 \alpha(R_\Delta/S_i)^3 \]