Distribution Functions of Dark Matter Halos

ASTR:6782
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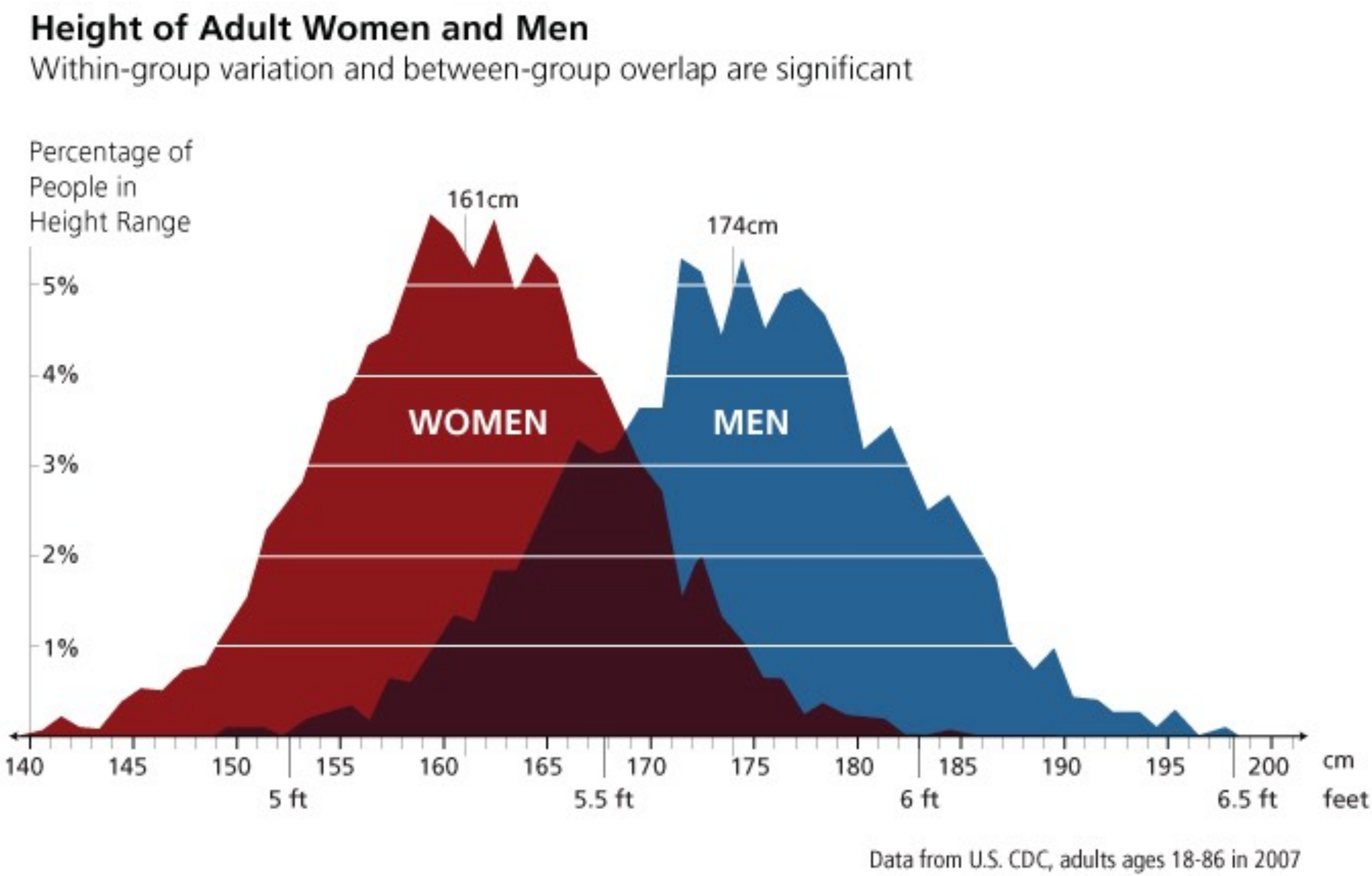
Statistical Properties of DM Halos:
Halo Mass Distributions predicted by
Press-Schechter 1974 Formalism

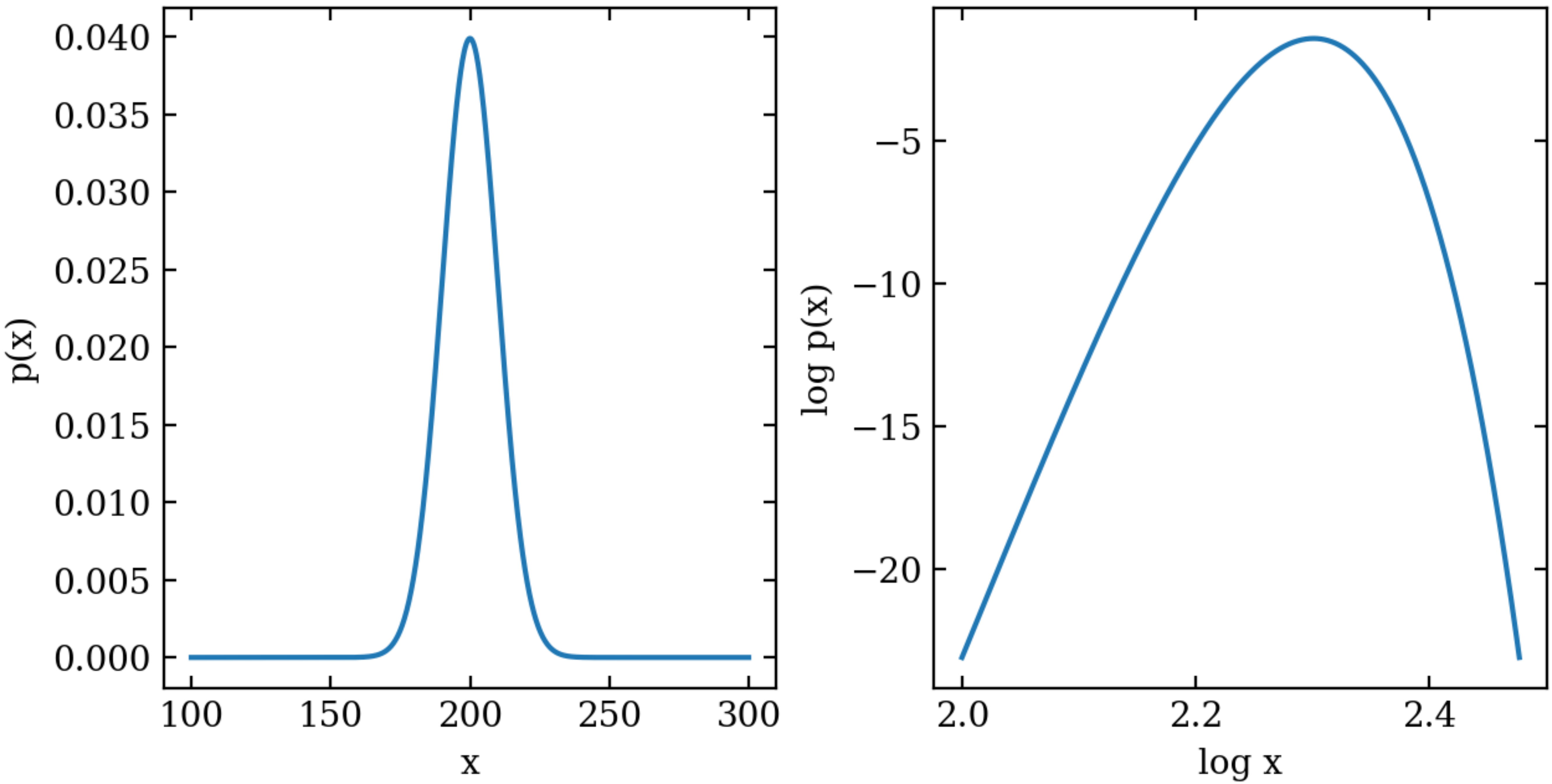
Distribution Functions in Everyday Life:

Gaussian and Power-law

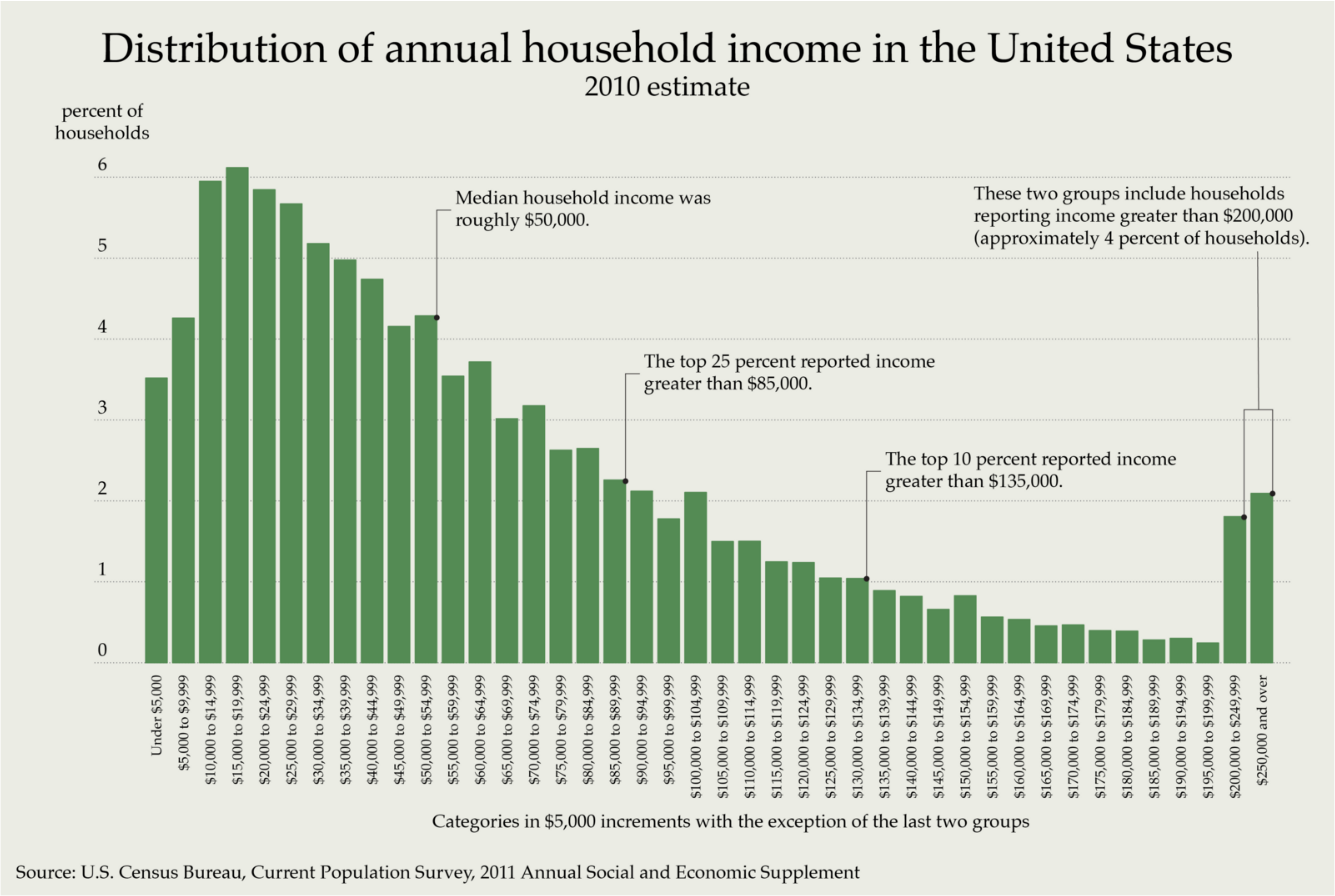
Gaussian Distribution Example: Height Distribution of Adults



Gaussian Distribution in Linear vs. Logarithmic Scales

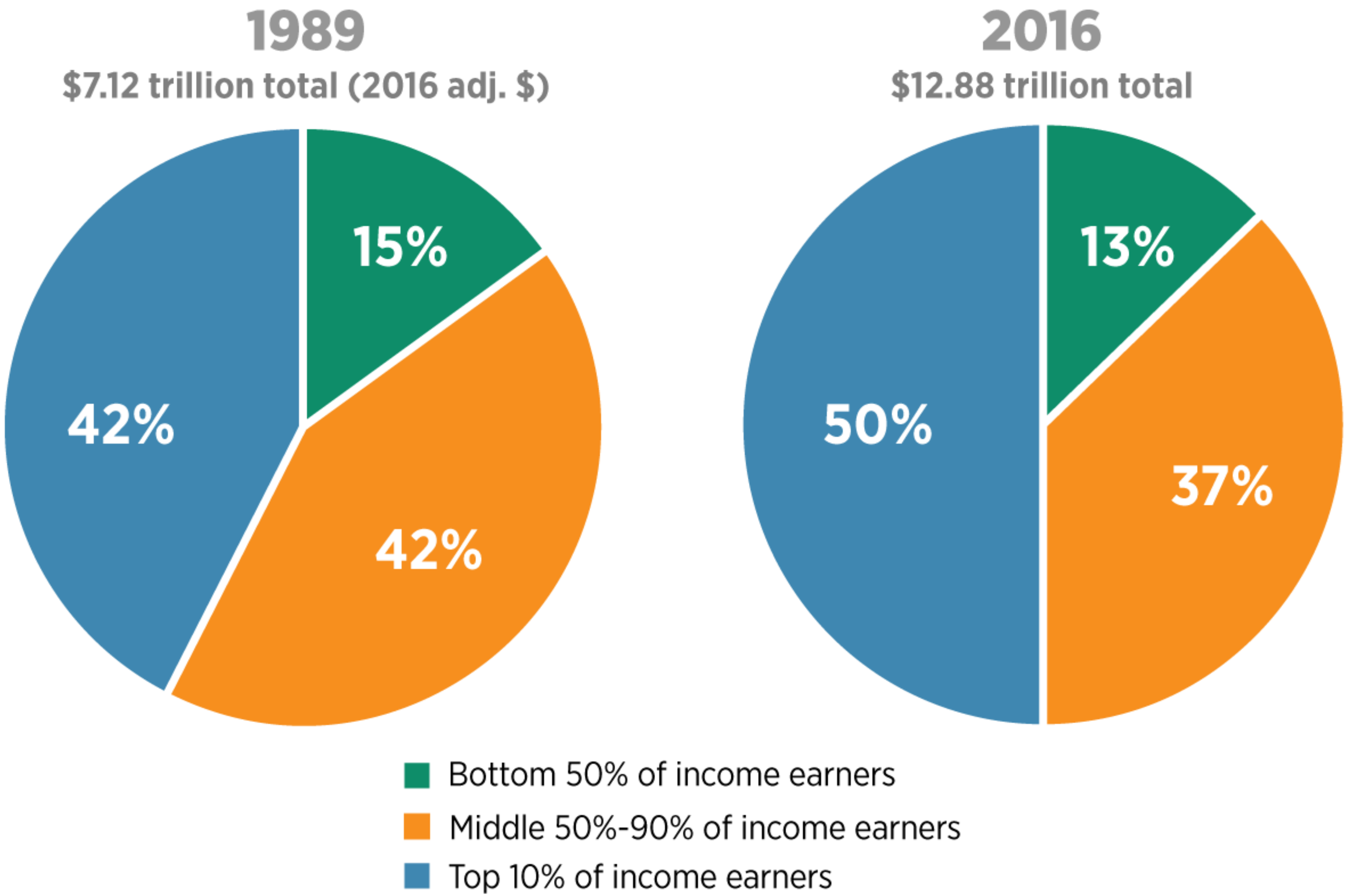


Power-law Distribution Example: Household Income Distribution in the US



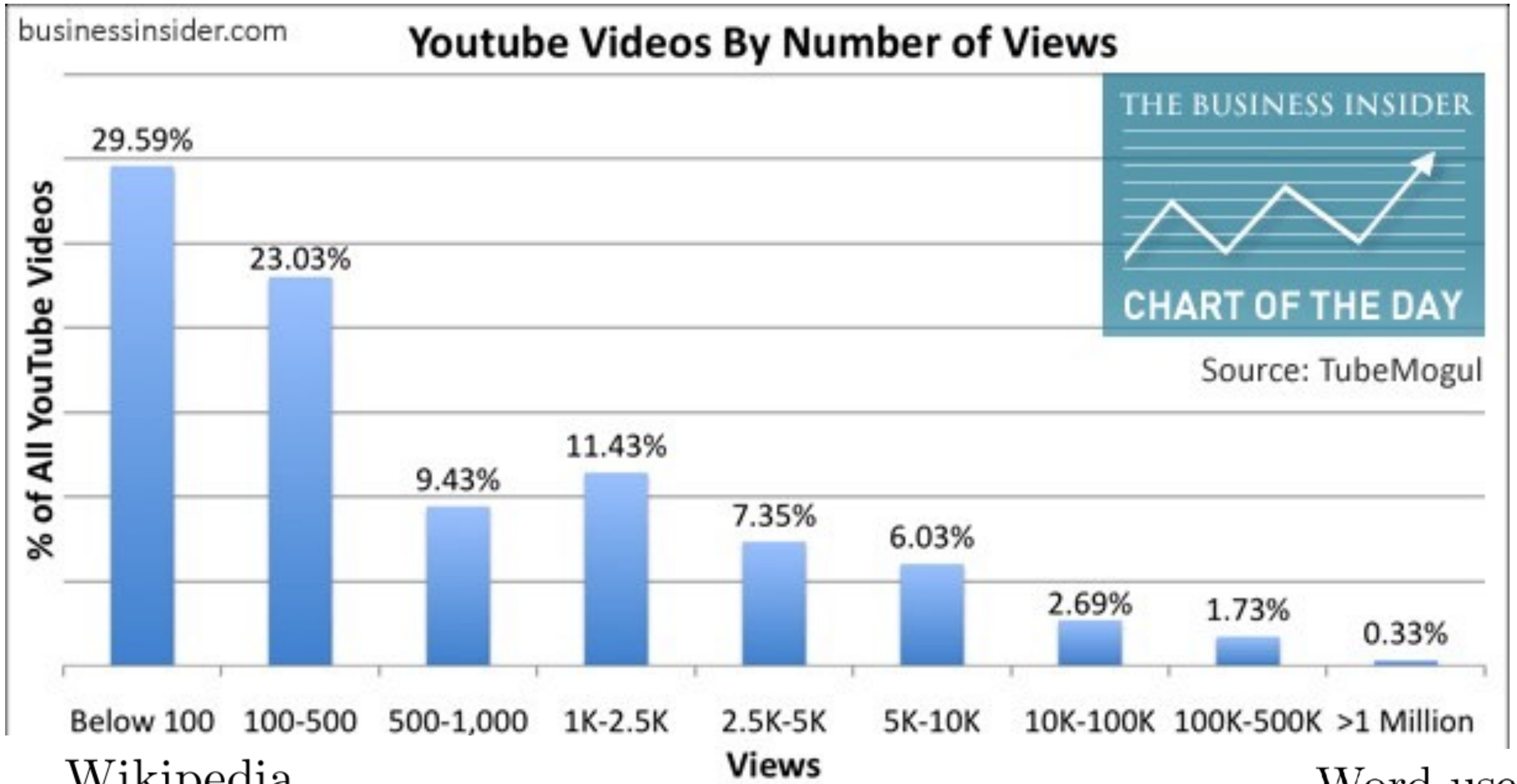
Income Distribution (Pre-tax)

How U.S. earners' shares of the total household income pie have changed

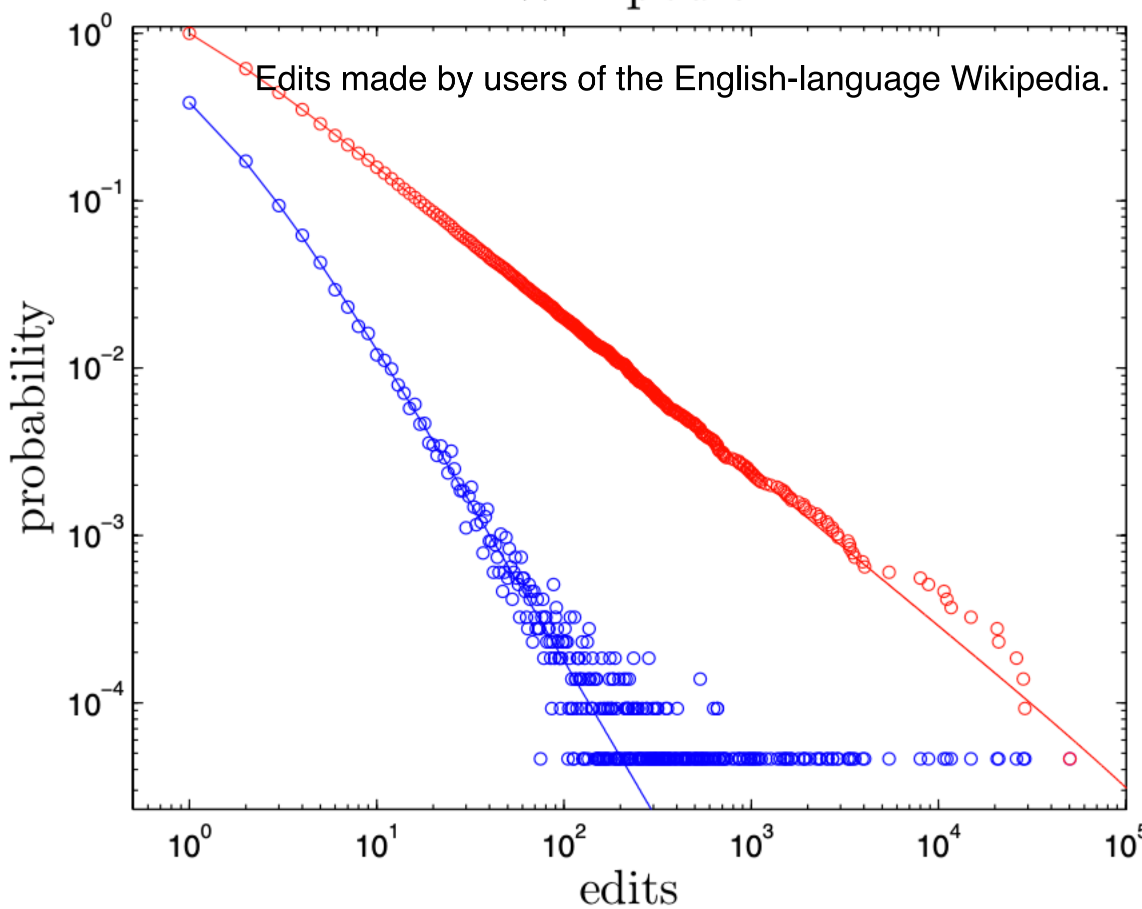


FEDERAL RESERVE BANK OF ST. LOUIS

More Power-law Distribution Examples

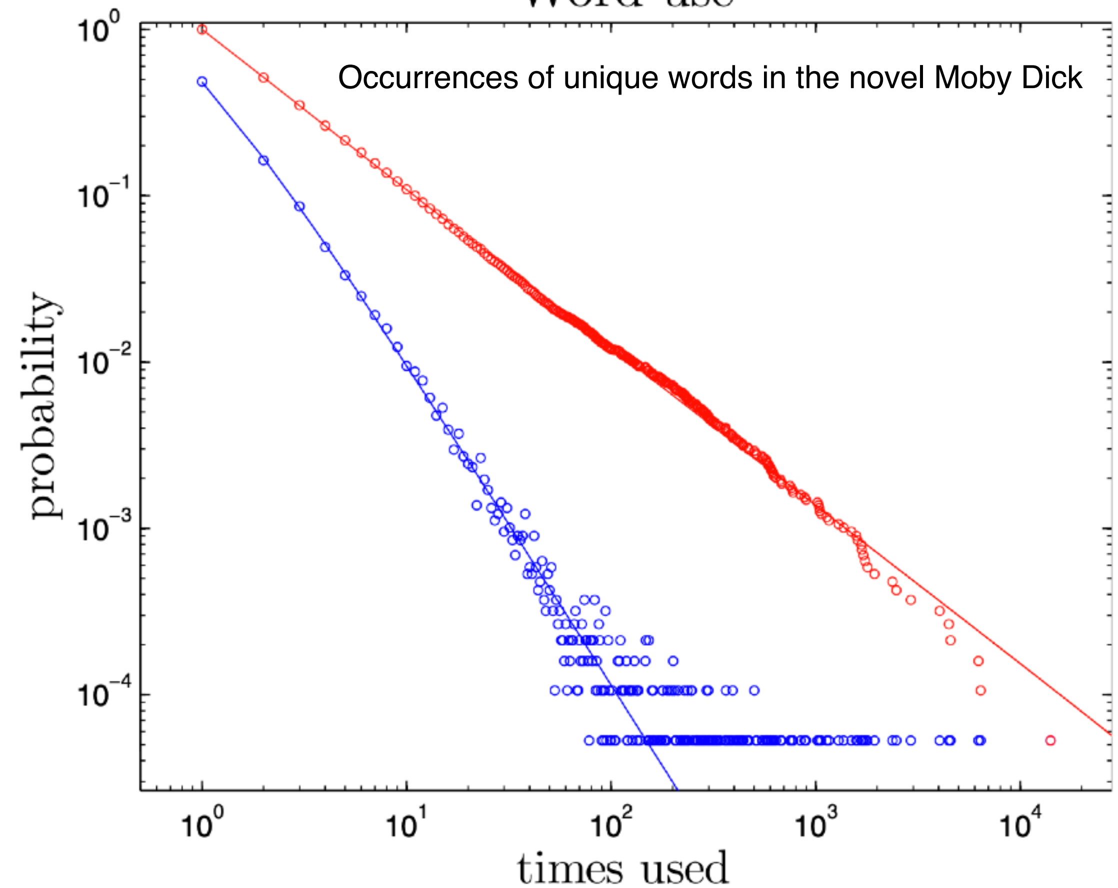


Wikipedia



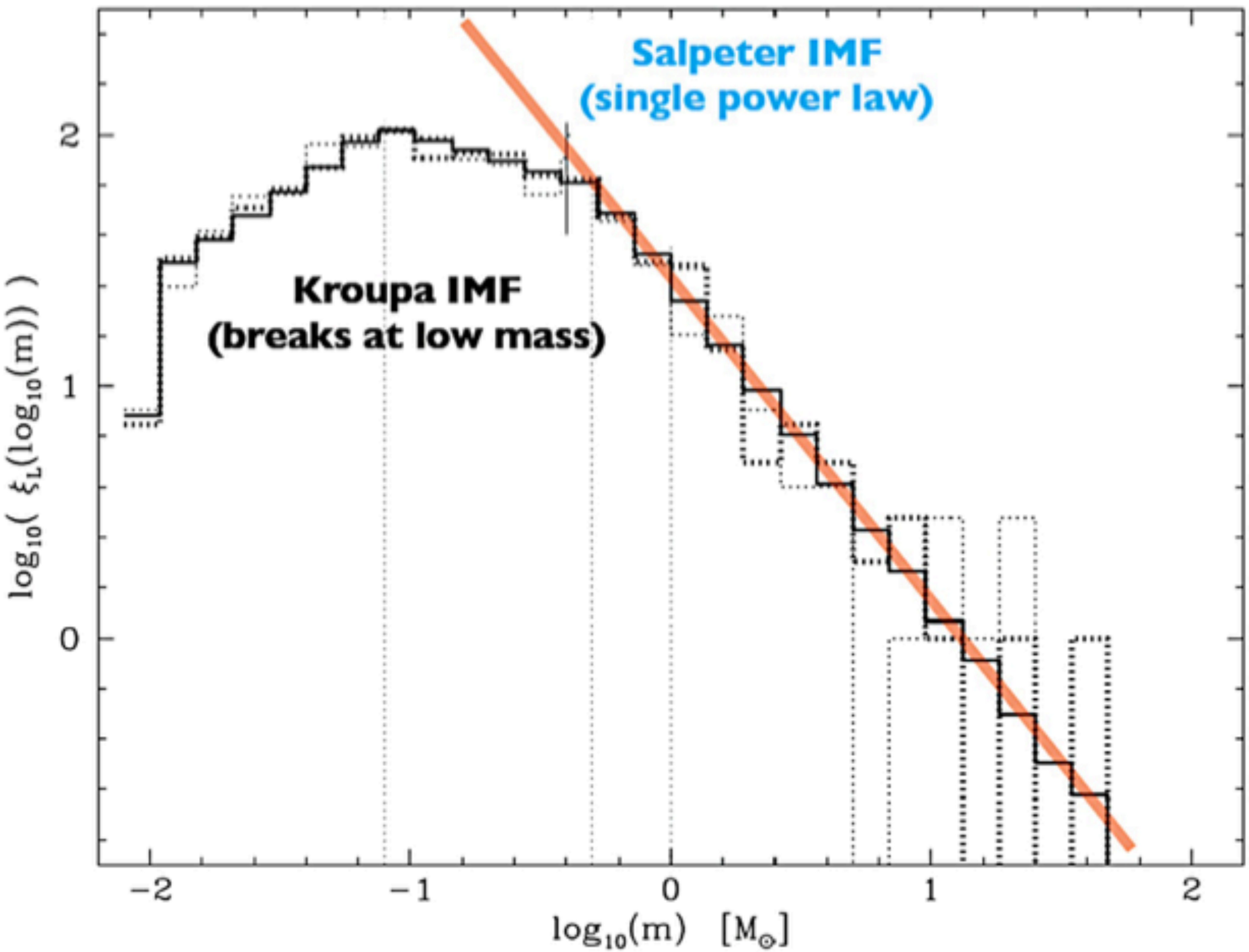
Word use

Peterson+2013 PNAS

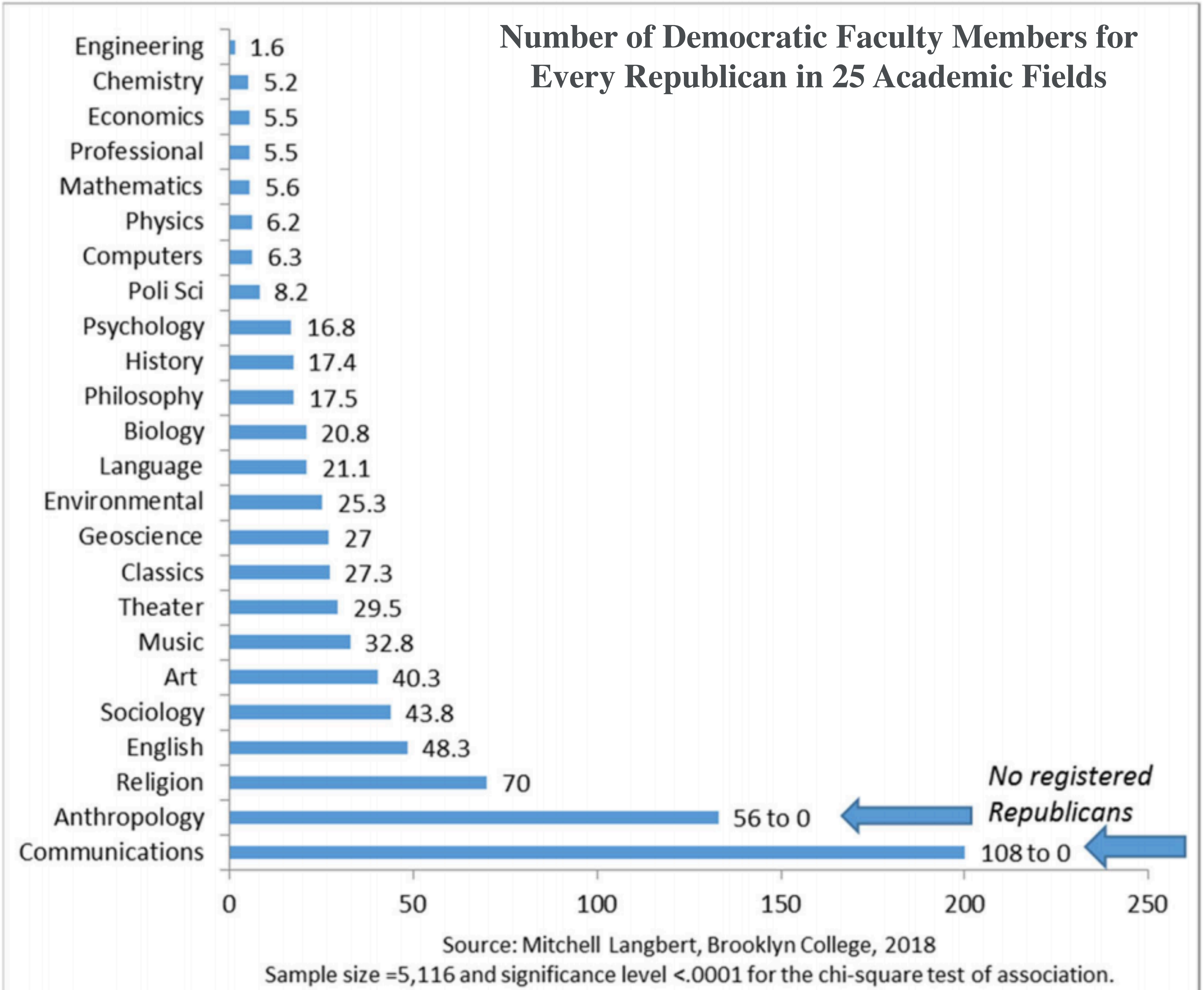


Power-Law Distribution of Stellar Masses in zero-age Clusters

$$\xi_L(m) = \frac{dN}{d\log m} = \ln(10)\ m\ \frac{dN}{dm} = \ln(10)\ m\ \xi(m)$$



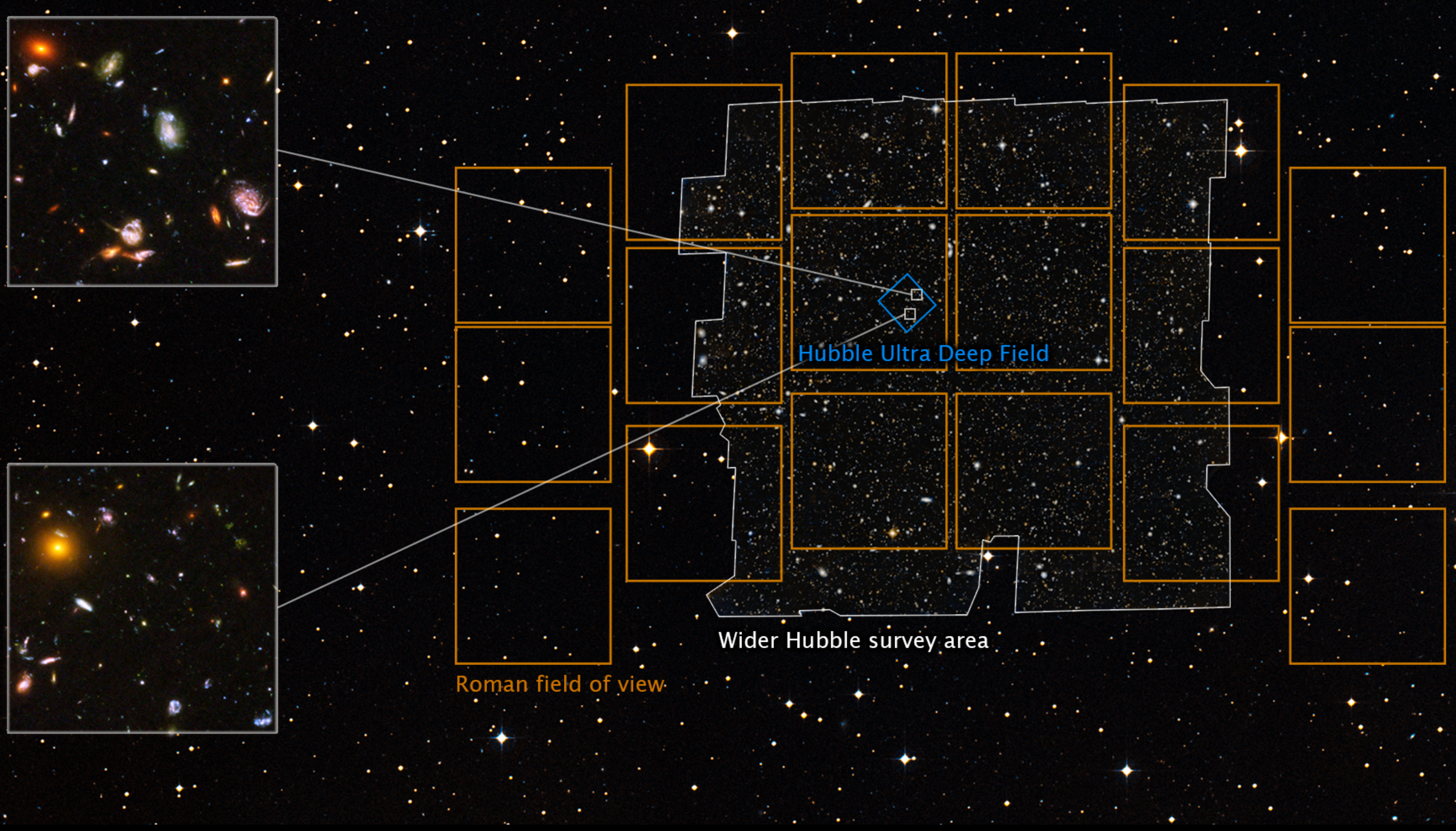
Examples of Biased Samples



https://www.nas.org/academic-questions/31/2/homogenous_the_political_affiliations_of_elite_liberal_arts_college_faculty

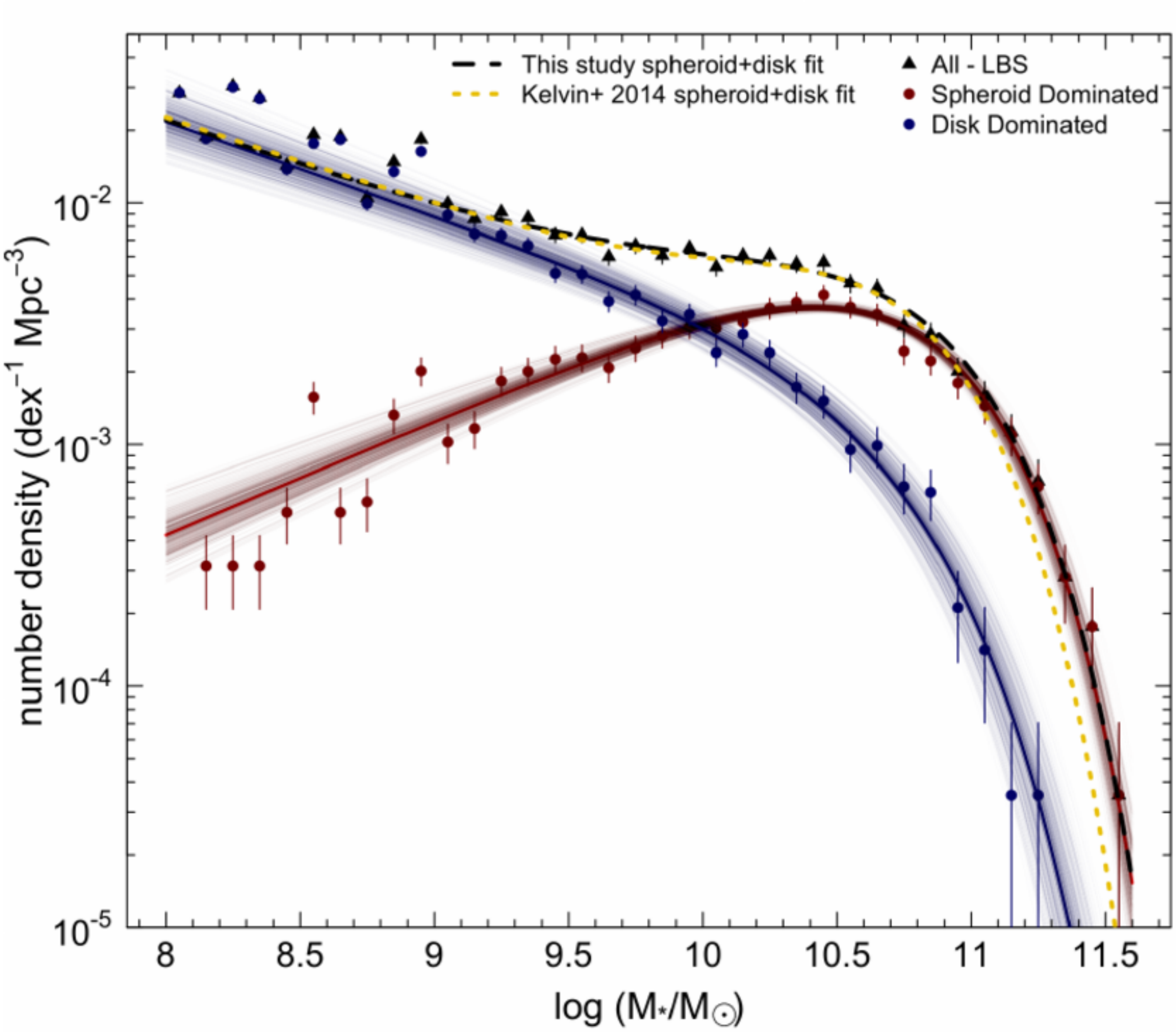
How Galaxies Distribute in Luminosity and Mass?

Power-law Decline x Exponential Decline

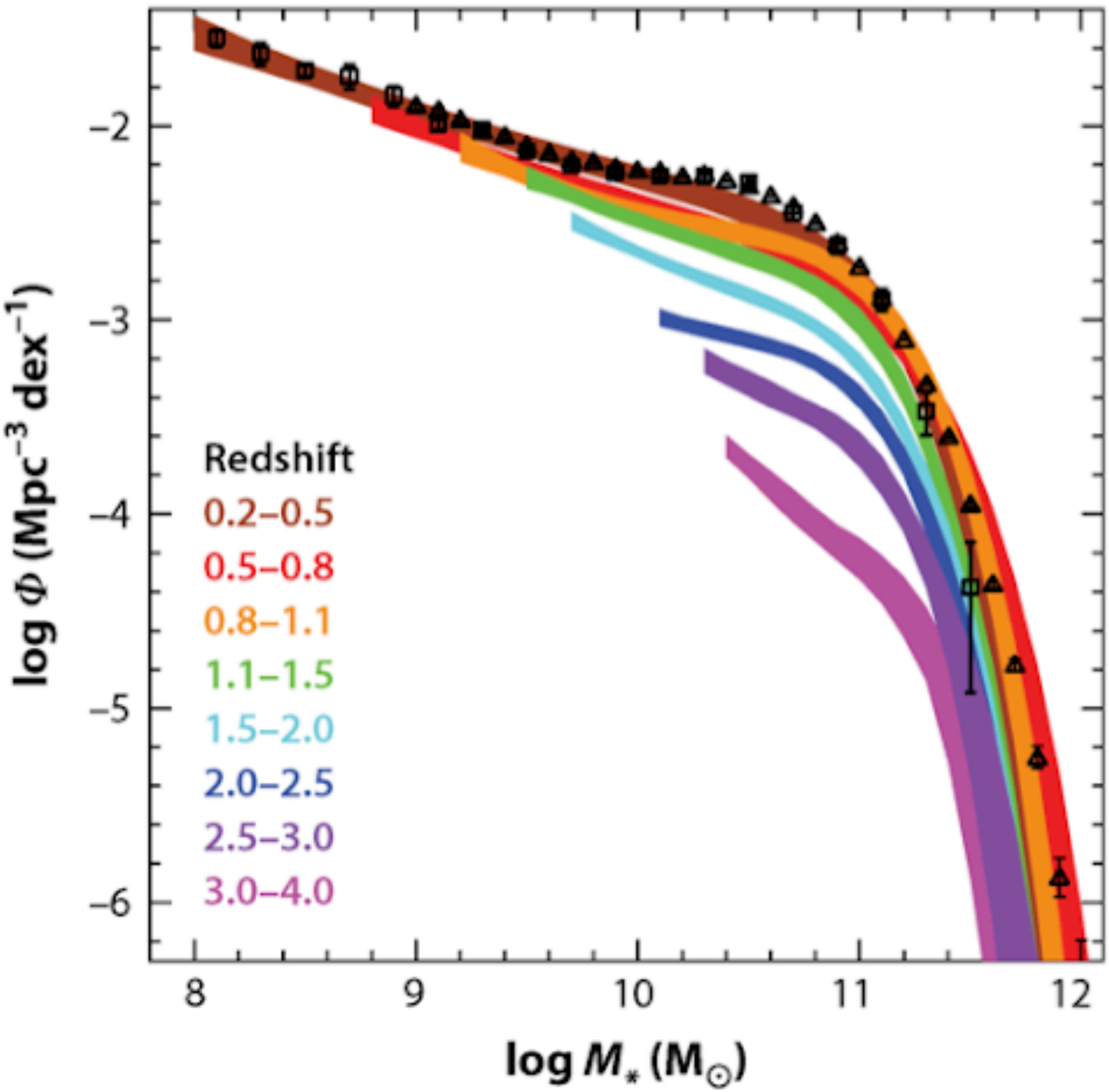


The Mass Distribution Functions of Disk Galaxies vs. Elliptical Galaxies

Moffett et al. 2015: stellar mass function at z ~ 0.05 (D ~ 200 Mpc)



The Evolution of the Mass Distribution Functions of Galaxies



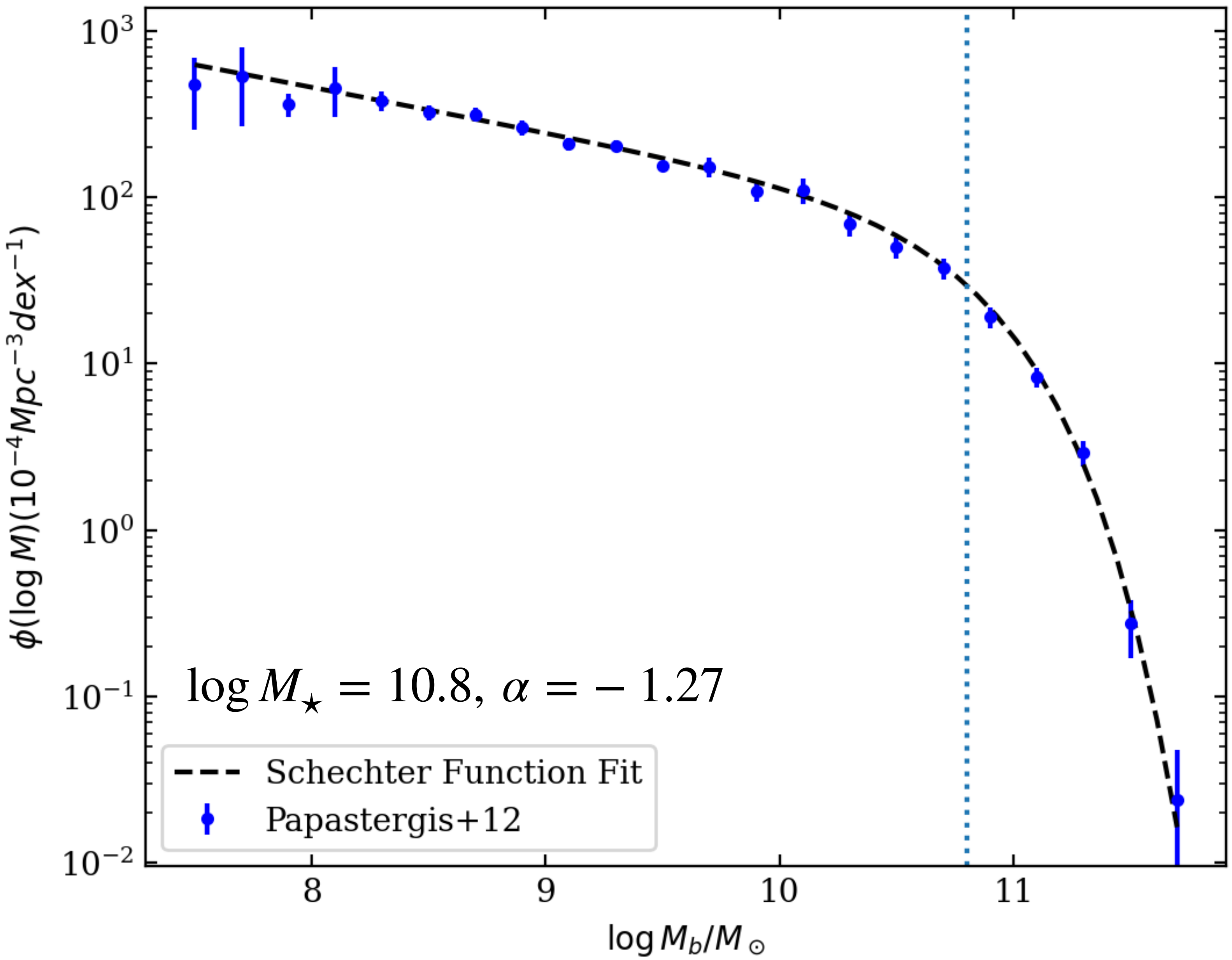
Schechter (1976) Function Fit to the Observed Distributions

Linear form: $$\phi(L) = \frac{\mathrm{d}N}{\mathrm{d}V\mathrm{d}L} = \frac{\phi_\star}{L_\star}\left(\frac{L}{L_\star}\right)^{\alpha} \exp\left(-\frac{L}{L_\star}\right)$$

$\phi(L)dL$ gives the number density of galaxies with luminosities between L and L+dL.

Log form: $$\phi(\log L) = \frac{\mathrm{d}N}{\mathrm{d}V\mathrm{d}\log L} = \ln(10)\phi_* 10^{(\alpha+1)(\log L - \log L_\star)} \exp(-10^{\log L - \log L_\star})$$

$\phi(\log L)d\log L$ gives the number density of galaxies with luminosities between logL and logL+dlogL.



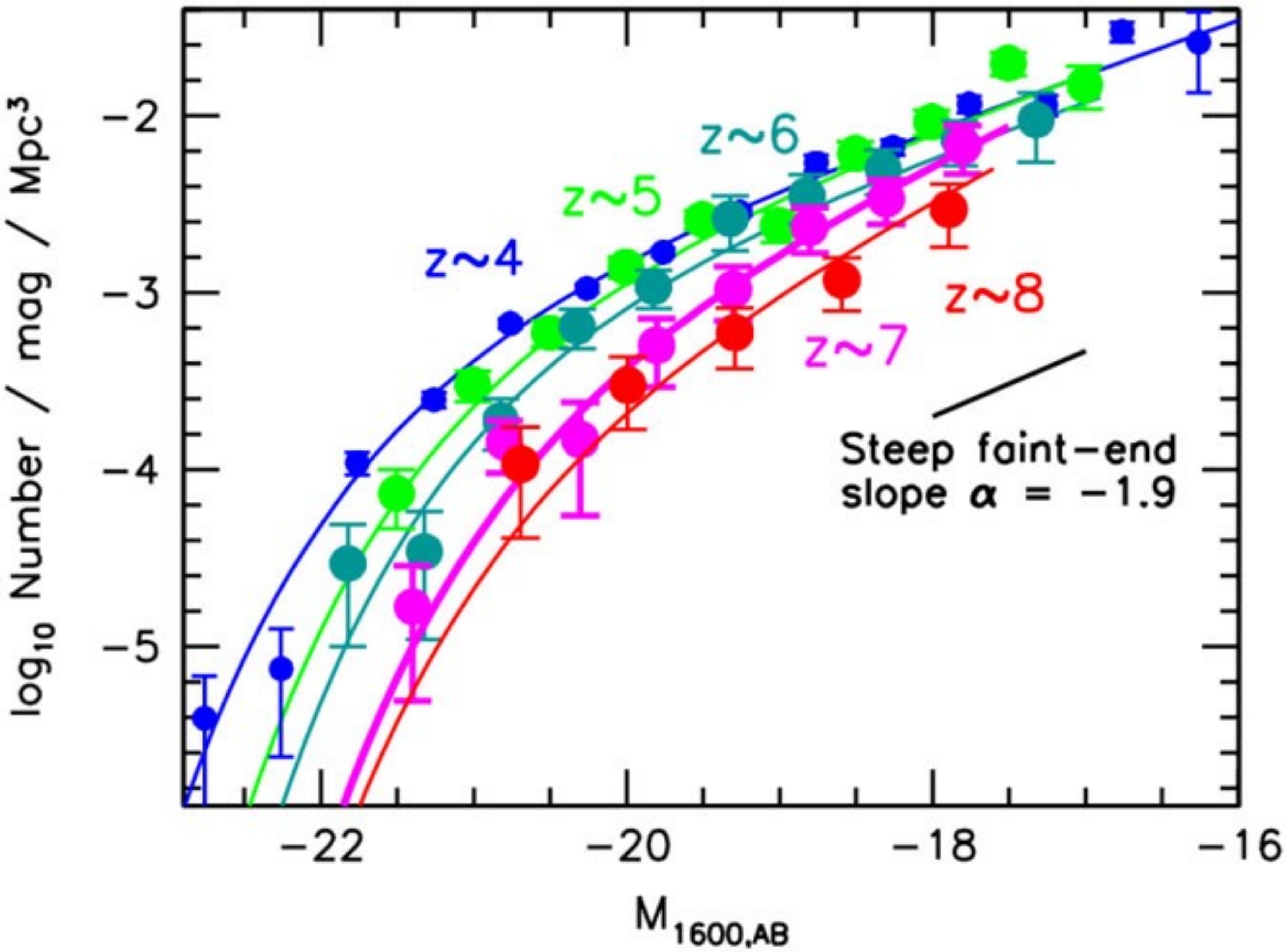
Schechter (1976) Function Fit to the Observed Distributions

Magnitude form:

$$\phi(M) = \frac{\mathrm{d}N}{\mathrm{d}V\mathrm{d}M} = (0.4\ln 10)\phi_* 10^{-0.4(M-M_*)(\alpha+1)} \exp\left[-10^{-0.4(M-M_*)}\right]$$

Log form:

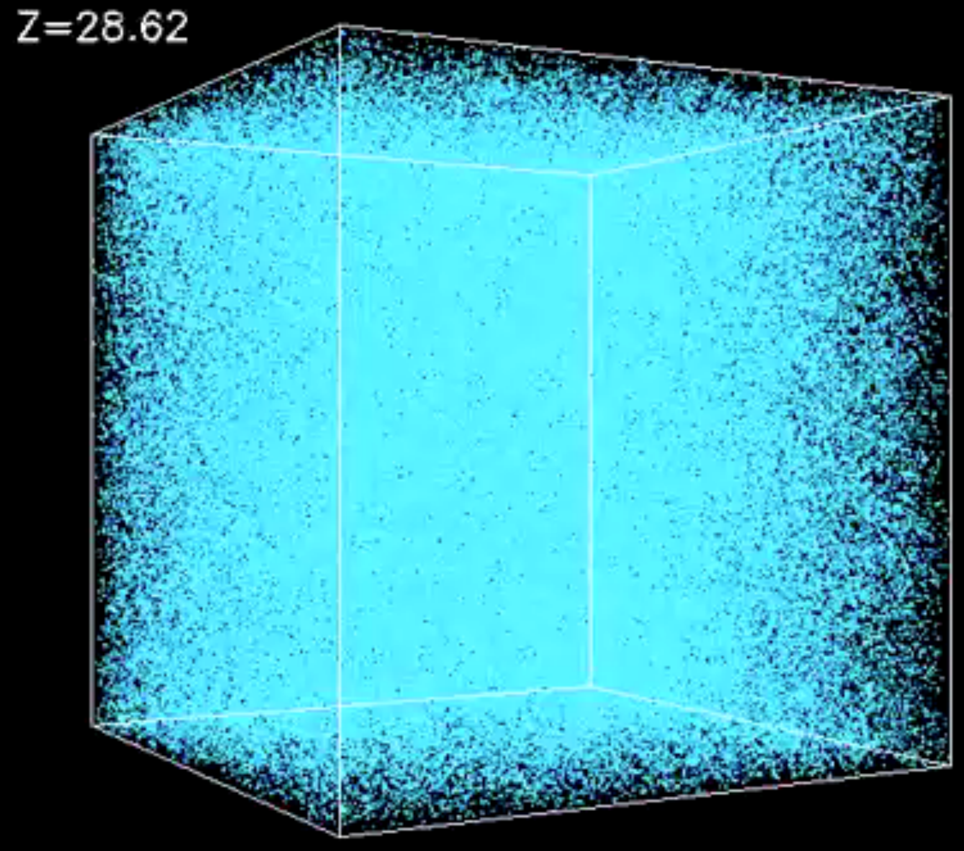
$$\phi(\log L) = \frac{\mathrm{d}N}{\mathrm{d}V\mathrm{d}\log L} = \ln(10)\phi_* 10^{(\alpha+1)(\log L - \log L_\star)} \exp\left(-10^{\log L - \log L_\star}\right)$$



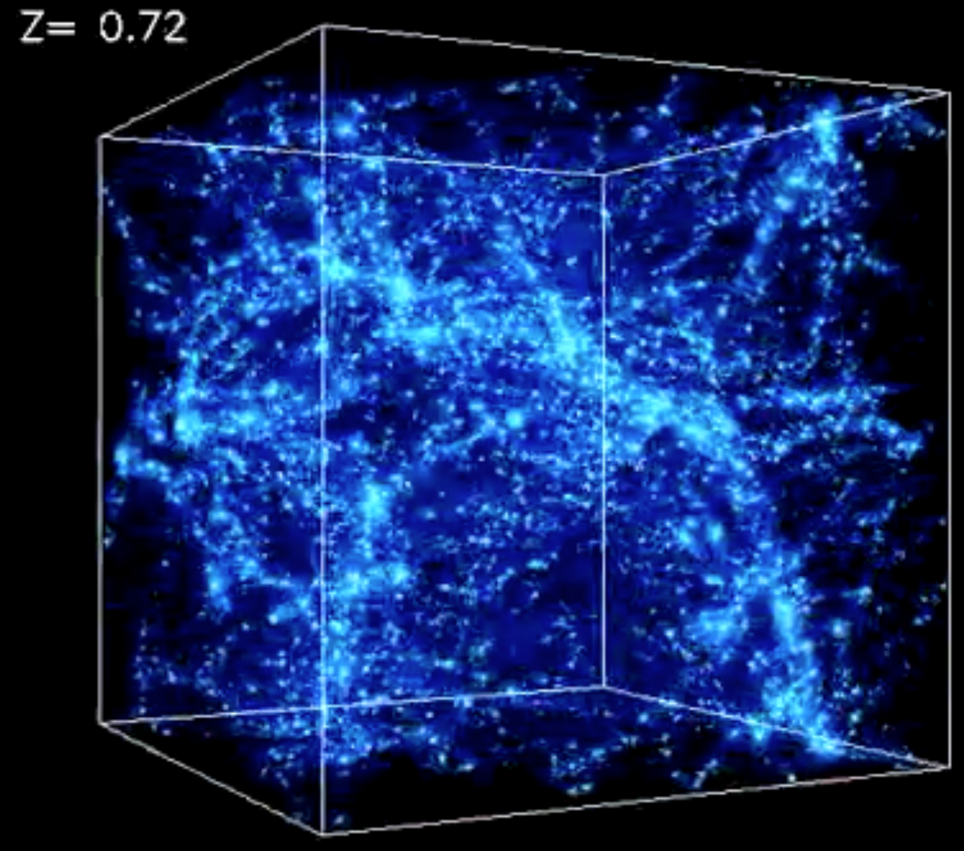
Motivation: Could the observed distributions of galaxies' luminosities (masses) originate from random density fluctuations?

Since the **gravitational evolution is deterministic**, we should be able to predict the statistical distributions of DM halos based on the initial density fluctuations, instead of running N-body simulations

Initial density fluctuations



N-body simulation of a comoving volume that is 40 Mpc across



THE ASTROPHYSICAL JOURNAL, **187**:425–438, 1974 February 1

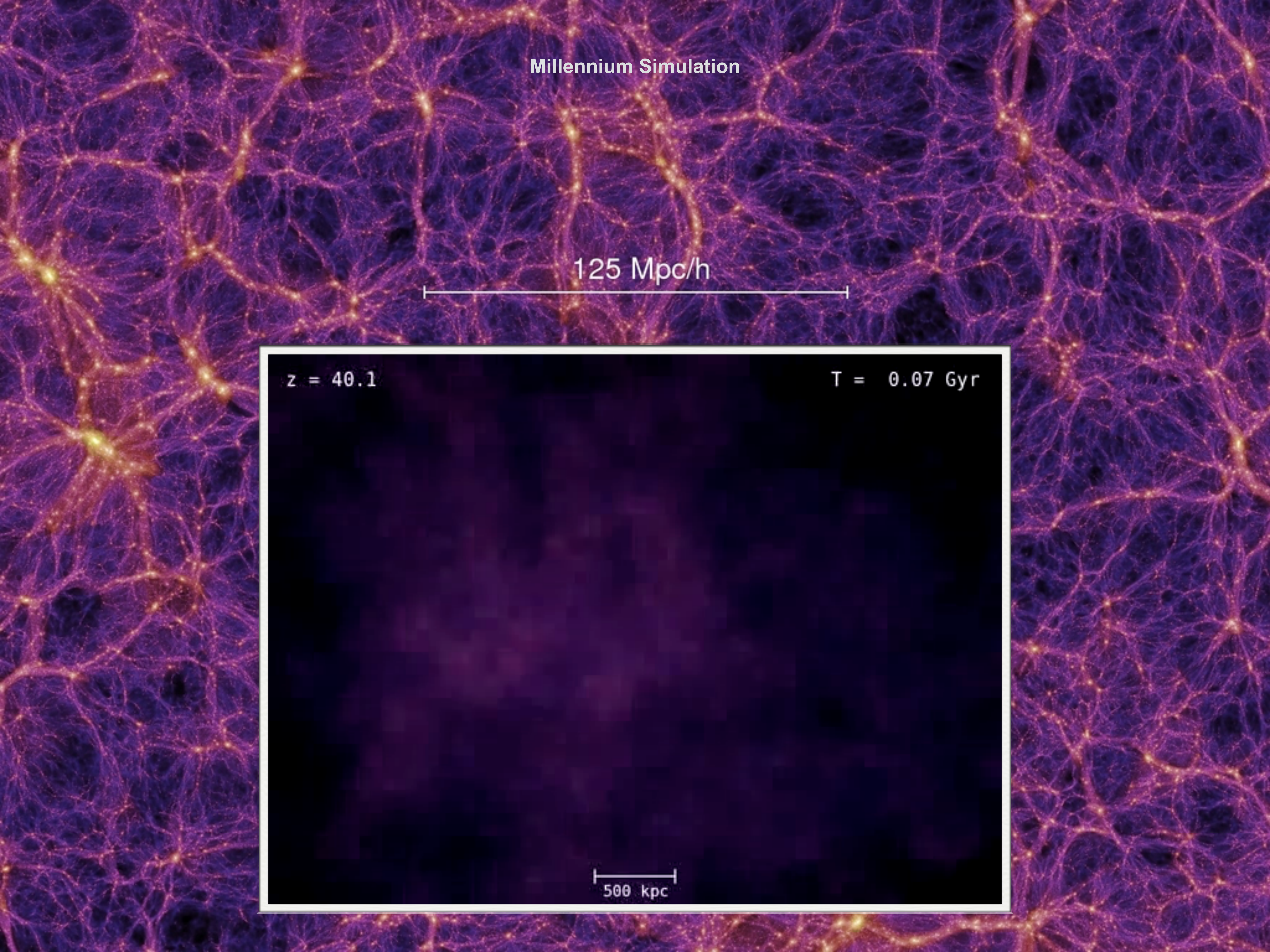
FORMATION OF GALAXIES AND CLUSTERS OF GALAXIES BY SELF-SIMILAR GRAVITATIONAL CONDENSATION*

WILLIAM H. PRESS AND PAUL SCHECHTER
California Institute of Technology
Received 1973 August 1

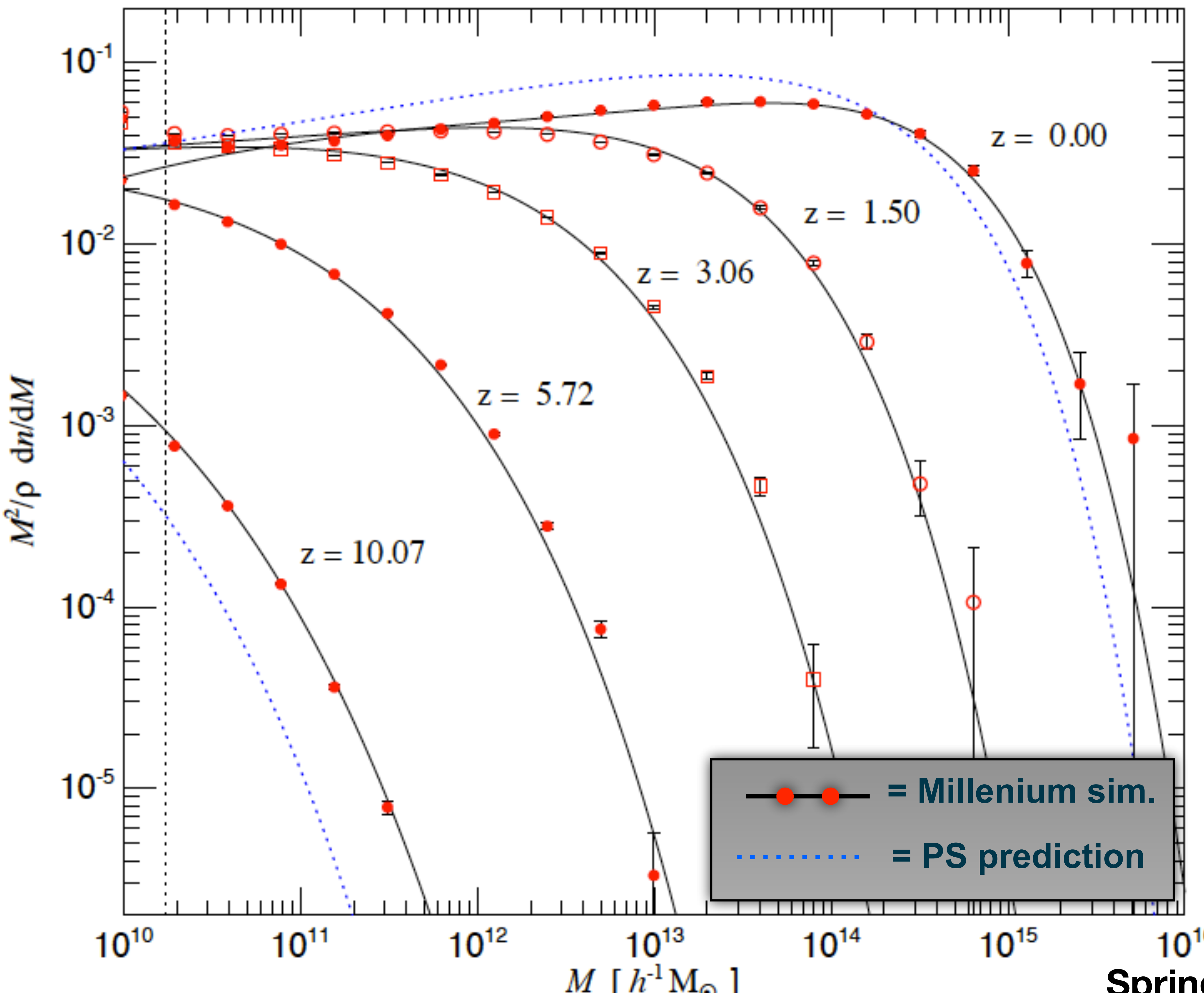
ABSTRACT

We consider an expanding Friedmann cosmology containing a "gas" of self-gravitating masses. The masses condense into aggregates which (when sufficiently bound) we identify as single particles of a larger mass. We propose that after this process has proceeded through several scales, the mass spectrum of condensations becomes "self-similar" and independent of the spectrum initially assumed. Some details of the self-similar distribution, and its evolution in time, can be calculated with the linear perturbation theory. Unlike other authors, we make no ad hoc assumptions about the spectrum of long-wavelength initial perturbations: the nonlinear N-body interactions of the mass points randomize their positions and generate a perturbation to all larger scales; this should fix the self-similar distribution almost uniquely. The results of numerical experiments on 1000 bodies are presented; these appear to show new nonlinear effects: condensations can "bootstrap" their way up in size faster than the linear theory predicts. Our self-similar model predicts relations between the masses and radii of galaxies and clusters of galaxies, as well as their mass spectra. We compare the predictions with available data, and find some rather striking agreements. If the model is to explain galaxies, then isothermal "seed" masses of $\sim 3 \times 10^7 M_\odot$ must have existed at recombination. To explain clusters of galaxies, the only necessary seeds are the galaxies themselves. The size of clusters determines, in principle, the deceleration parameter q_0; presently available data give only very broad limits, unfortunately.

Subject headings: cosmology — galaxies — galaxies, clusters of


Millennium Simulation
125 Mpc/h
z = 40.1
T = 0.07 Gyr
500 kpc

Halo Mass Functions: N-body Simulation vs. Analytical Predictions



Springel+2005

Press & Schechter (1974)

Formalism

The objective is to estimate halo mass distributions

Based on the results from the **spherical collapse model**:

- Estimate the comoving volume density of collapsed halos more massive than M, at any z:

$$n_{halo}(> M, z)$$

cumulative mass function

- Estimate the comoving volume density of collapsed halos within a mass range of [M, M+dM], at any z:

$$\phi_{halo}(M, z)dM = \frac{dn_{halo}(> M, z)}{dM}dM$$

differential mass function

Cosmology: Einstein-de Sitter Universe

- Given the dimensionless Hubble parameter from FE1:

$$E(a) = \frac{H}{H_0} = \sqrt{(1-\Omega_0)/a^2 + \Omega_{m,0}/a^3 + \Omega_{\gamma,0}/a^4 + \Omega_{\Lambda,0}}$$

- and the rearranged time-scale factor relation:

$$\frac{t}{t_H} = \int_0^{1/(1+z)} \frac{da}{E(a)a} = \int_z^{\infty} \frac{dz'}{(1+z')E(z')}$$

- A **matter-only, flat** universe is known as the **Einstein-de Sitter** universe. It has the following density parameters:

$$\Omega_0 = \Omega_{m,0} = 1,\ \Omega_{\gamma,0} = \Omega_{\Lambda,0} = 0$$

which lead to the following analytical solution:

$$\Rightarrow H = H_0 a^{-3/2}$$

$$\Rightarrow t = \frac{2}{3} t_H a^{3/2} = \frac{2}{3} t_H (1+z)^{-3/2}$$

$$\Rightarrow \rho_c = \frac{3H^2}{8\pi G} = \rho_{c,0} a^{-3} = \frac{1}{6\pi G t^2}$$

PS74 Part I: Linear Growth & Collapse Threshold

Parameterized Solution of Top-Hat Spherical Collapse Model

- **initial scale factor** (inside=outside): $a_i = 1/z_{\rm recombination} \approx 10^{-3}$
- **radius of Top Hat** = **inside scale factor** x comoving radius

$$\frac{a(\theta)}{a_i} = \frac{1+\delta_i}{\delta_i}(1+\cos\theta) = A(1+\cos\theta)$$

- **time**

$$t(\theta) = \frac{1+\delta_i}{2H_i\delta_i^{3/2}}(\theta - \sin\theta) = B(\theta - \sin\theta)$$

since $H = H_0 a^{-3/2}$ in **EdS** Universe, $H_i = H_0 a_i^{-3/2}$, we obtain:

$$\frac{t(\theta)}{1/H_0} = \frac{1+\delta_i}{2a_i^{-3/2}\delta_i^{3/2}}(\theta - \sin\theta)$$

- **density contrast**:

$$1+\delta(\theta) = \frac{\rho}{\rho_c} = \frac{9}{2}\frac{(\theta - \sin\theta)^2}{(1-\cos\theta)^3}$$

at turn-around ($\theta = \pi$): $1+\delta_{ta} = 9\pi^2/16$

at virialization: $a_v = a_{ta}/2,\ t_v = 2t_{ta} \Rightarrow 1+\delta_v = 2^5(1+\delta_{ta}) = 18\pi^2$

The Linear Theory: Simplifying Solutions by Taylor Expansion

- starting from the **density contrast = 1 + overdensity**:

$$1 + \delta(\theta) = \frac{\rho}{\rho_c} = \frac{9}{2}\frac{(\theta - \sin\theta)^2}{(1 - \cos\theta)^3}$$

- when $\theta \ll 1$, we can use **Taylor expansions** to show:

$$1 + \delta(\theta) \approx 1 + \frac{3}{20}\theta^2$$

- when $\theta \ll 1$, we can also express θ as a function of time

$$t = B(\theta - \sin\theta) \approx B\theta^3/6, \text{ so } \theta = \left(\frac{6t}{B}\right)^{1/3} = \left(\frac{6\pi t}{t_{ta}}\right)^{1/3} \text{ where } t_{ta} = \pi B$$

- combining the results, we have **overdensity** as function of time:

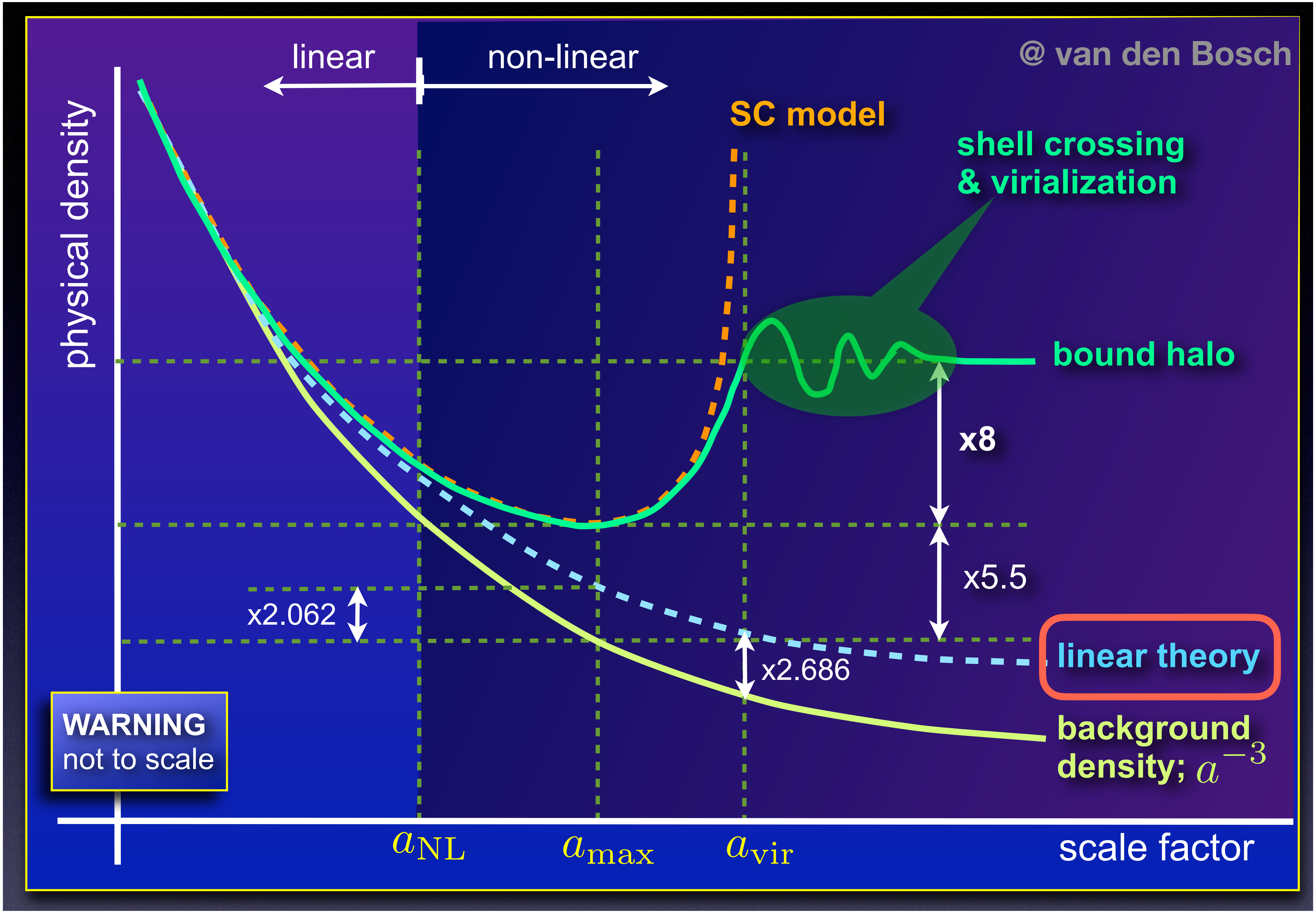
$$\delta(t) \approx \frac{3}{20}\left(\frac{6\pi t}{t_{ta}}\right)^{2/3}$$

which equals **1.062 at turnaround ($t = t_{ta}$)** and **1.686 at virialization ($t = 2t_{ta}$)**

- **EdS:** the **outside** scale factor is

$$a(t) = \left(\frac{3t}{2t_H}\right)^{2/3}$$, so that $\delta \propto a = 1/(1+z)$ **(Linear growth of overdensity)**

Density Contrast: Non-Linear vs. Linear Growth



Halo mass: non-linear vs. linear growth

- The top-hat spherical collapse model motivated the definition of the virial radius and virial mass of a collapsed object:

$$\bar{\rho}(r < r_\Delta) = \Delta_c \rho_c \text{ and } M_\Delta = \frac{4\pi}{3} r_\Delta^3 \Delta_c \rho_c \text{ where}$$

$$\Delta_c = 200 \approx 18\pi^2$$

- For **linear growth theory**, the mass of the collapsed object is simply the cosmic mean density multiplied by the physical scale *R within which overdensities are expected to collapse to form a single object*:

$$M(R) = \frac{4}{3}\pi R^3 \rho_c(z) \equiv \gamma \rho_c(z) R^3$$

The Linear Theory: Collapse Threshold at Each Redshift

- From linear extrapolation, we have two main results:
  - overdensity multiplied by *(1+z)* is a constant:
    $\delta \propto a = 1/(1+z) \Rightarrow \delta(1+z) = \text{const.}$
  - overdensity at virialization is a constant: $\delta_v = 1.686$
- So, any **linearly extrapolated overdensities** reaching 1.686 at a given z should have collapsed and virialized at z. Mathematically, this is:

$$\delta(t) = \delta(t_i)\frac{a(t)}{a(t_i)} = \frac{\delta(t_i)(1+z_i)}{1+z} > 1.686$$

  which can be rearranged as:

$$\delta(t_i)(1+z_i) > 1.686(1+z)$$

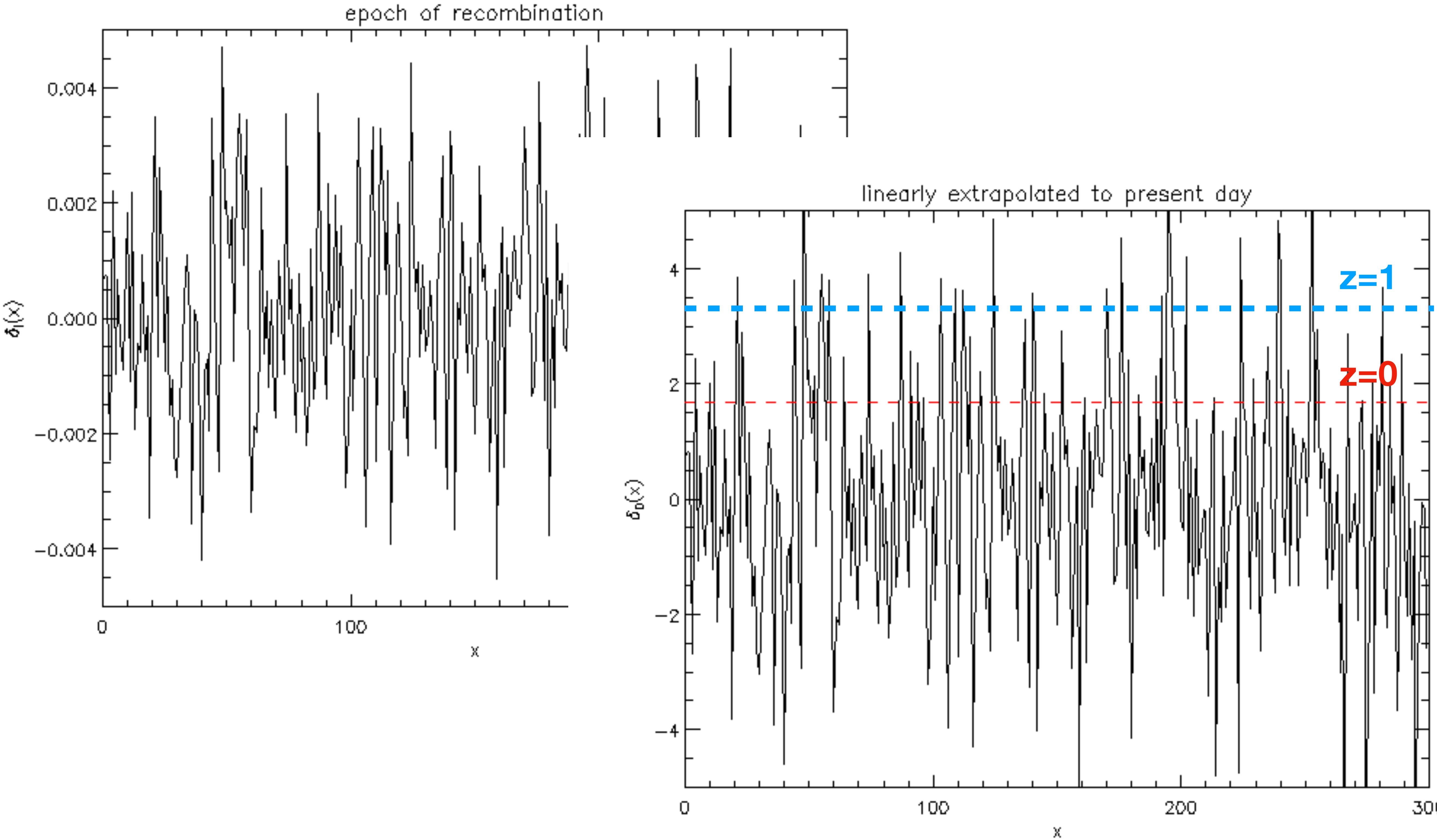
- **The left side** is the initial density perturbation field linearly extrapolated to today, since $\delta(t_i)(1+z_i) = \delta(t_0)$,
- **The right side** is a redshift-dependent threshold $\delta_c(z)$, above which the initial overdensity should have collapsed by redshift z. The higher the redshift, the greater the threshold.

$$\delta(t_i)(1+z_i) = \boxed{\delta(t_0) > \delta_c(z)} = 1.686(1+z)$$

The Criterion of Halo Formation at redshift z:

the *Initial* density perturbation field linearly extrapolated to *Today* ***exceeds the overdensity threshold for redshift z***

$$\delta(x; t_i)(1 + z_i) = \delta(x; t_0) > \delta_c(z) = 1.686(1 + z)$$

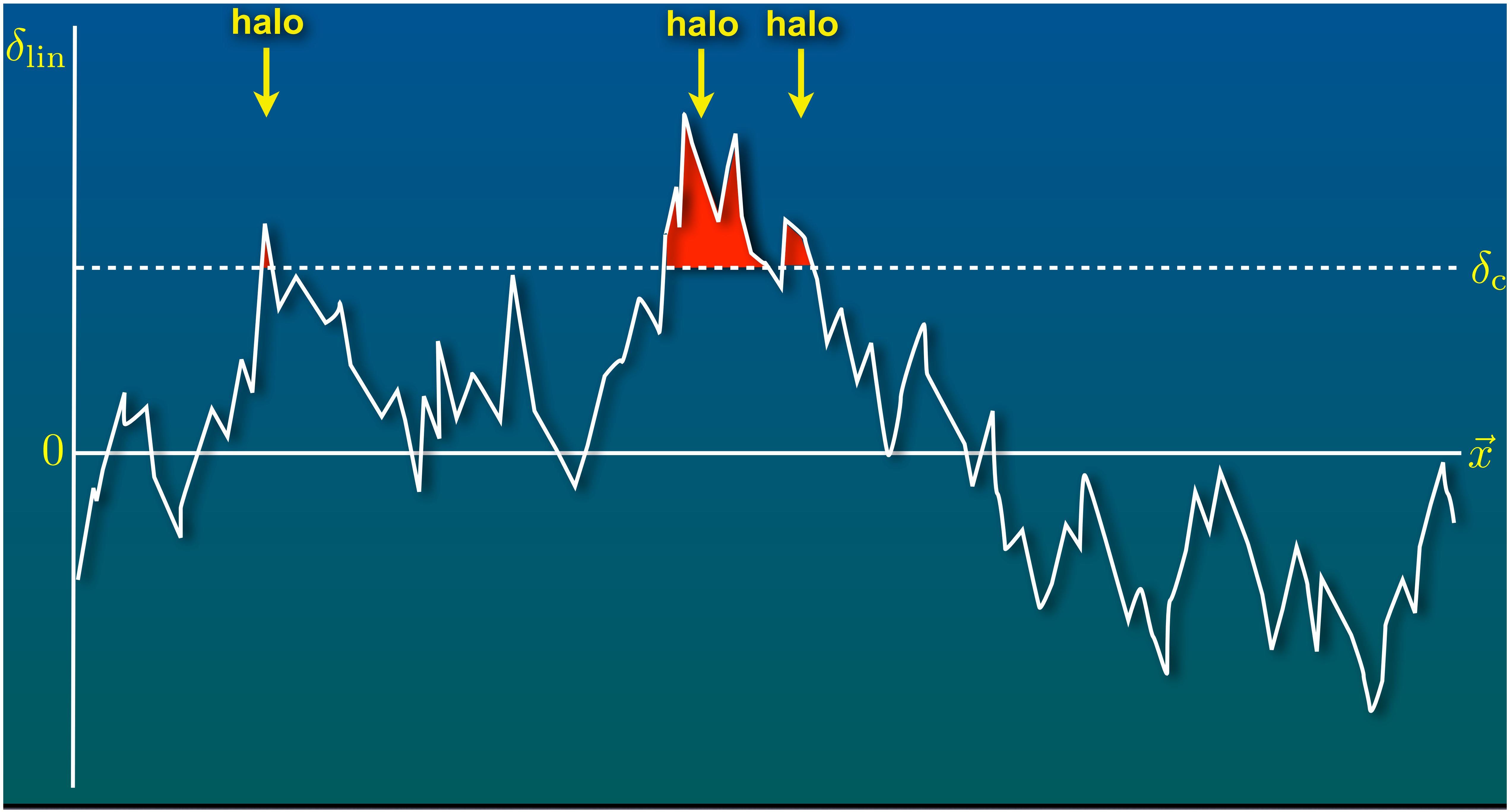


Note: the spatial axes are **comoving distances**

Counting the number of collapsed halos at each redshift

The total number of halos that have formed by redshift of z is:

$$N_{\text{halo}}(z) = N_{\text{cell}} P[\delta(t_0) > \delta_c(z)]; \text{ where } \delta_c(z) = 1.686\ (1+z)$$

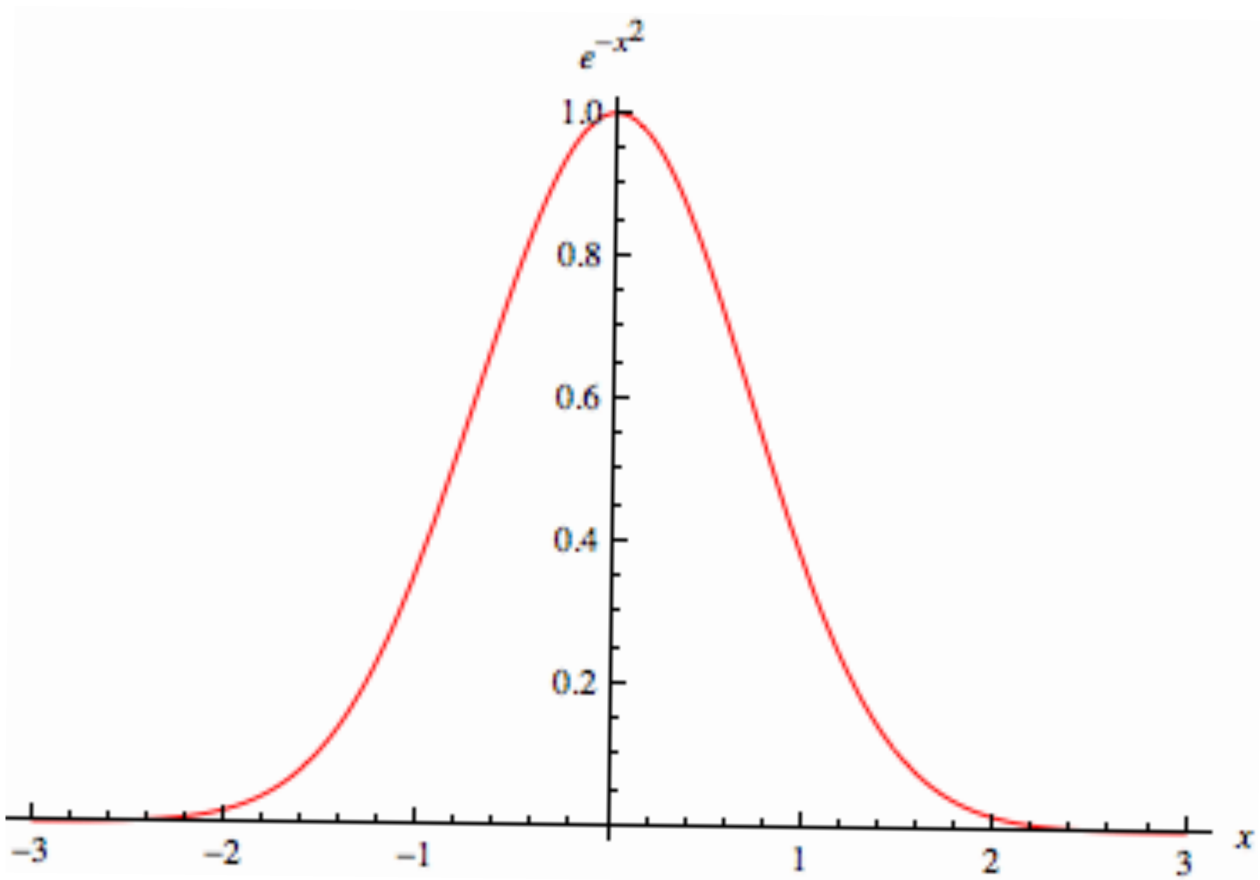


Random density fluctuations following a Gaussian distribution

For **Gaussian** random fields, the integral above a threshold is the **complementary error function**:

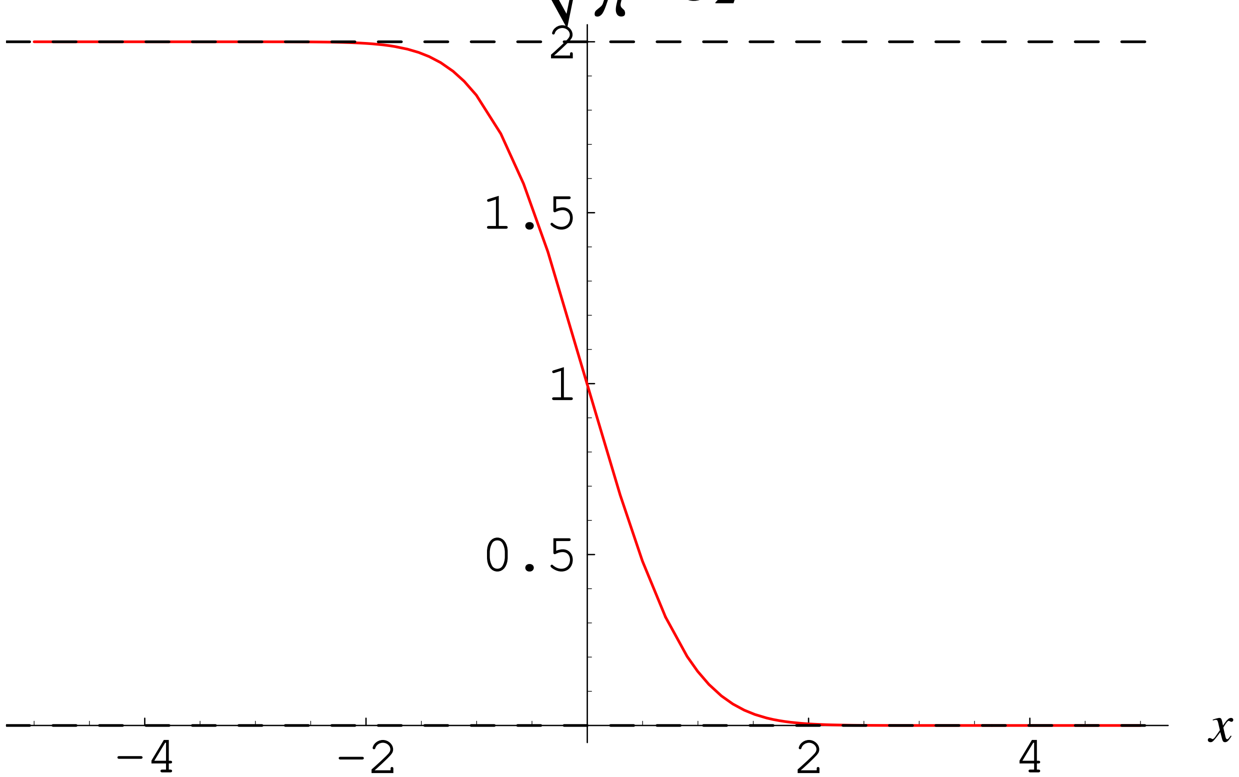
$$P(\delta > \delta_c) = \int_{\delta_c}^{\infty} \frac{1}{\sqrt{2\pi\sigma}} \exp(-\frac{\delta^2}{2\sigma^2}) d\delta = \frac{1}{2} - \frac{1}{2}\mathrm{erf}(\frac{\delta_c}{\sqrt{2}\sigma}) = \frac{1}{2}\mathrm{erfc}(\frac{\delta_c}{\sqrt{2}\sigma})$$

Gaussian function



complementary error function

$$\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^{\infty} e^{-t^2}\, dt.$$



The erfc function gives the number density of *all* collapsable regions, what is the **minimum mass** of these collapsed?

- For **linear growth theory**, the mass of the collapsed object is simply the cosmic mean density multiplied by the physical scale *R within which overdensities are expected to collapse to form a single object*:

$$M(R) = \frac{4}{3}\pi R^3 \rho_c(z) \equiv \gamma \rho_c(z) R^3$$

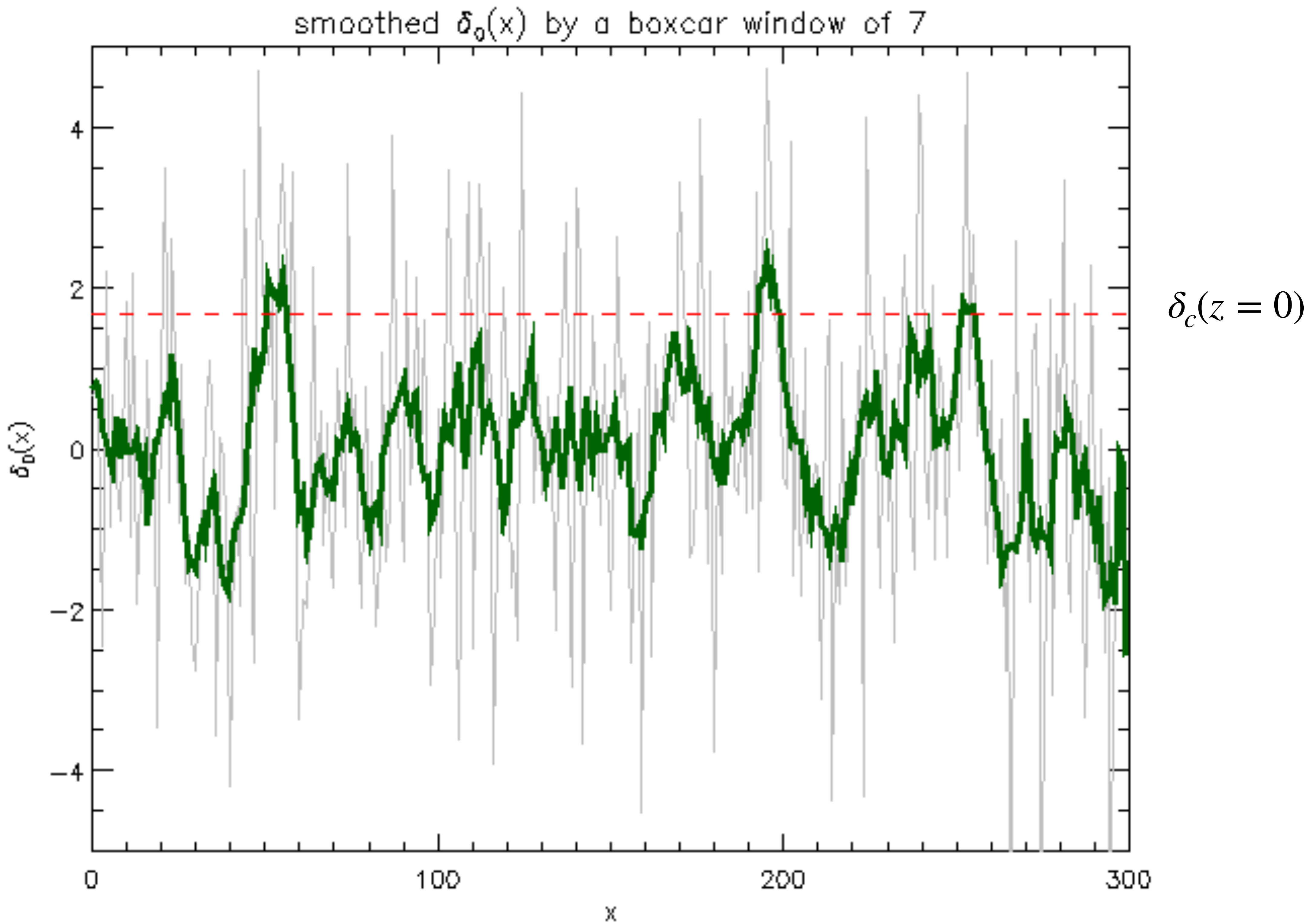
- The minimum mass is thus determined by the minimum physical scale, which is the spatial resolution of the density field.

- Note that if *R* is in **comoving** unit (cMpc), the critical density $\rho_c(z)$ must also be converted to $\rho_{c,0}$ **comoving** units (M_{sun} cMpc^{-3}), because by default $\rho_c(z) \propto a^{-3}$ is in physical units (M_{sun} Mpc^{-3}). As a result:

$$M(R) = \gamma \rho_{c,0} R^3 \text{ where } \rho_{c,0} = \frac{3H_0^2}{8\pi G}$$

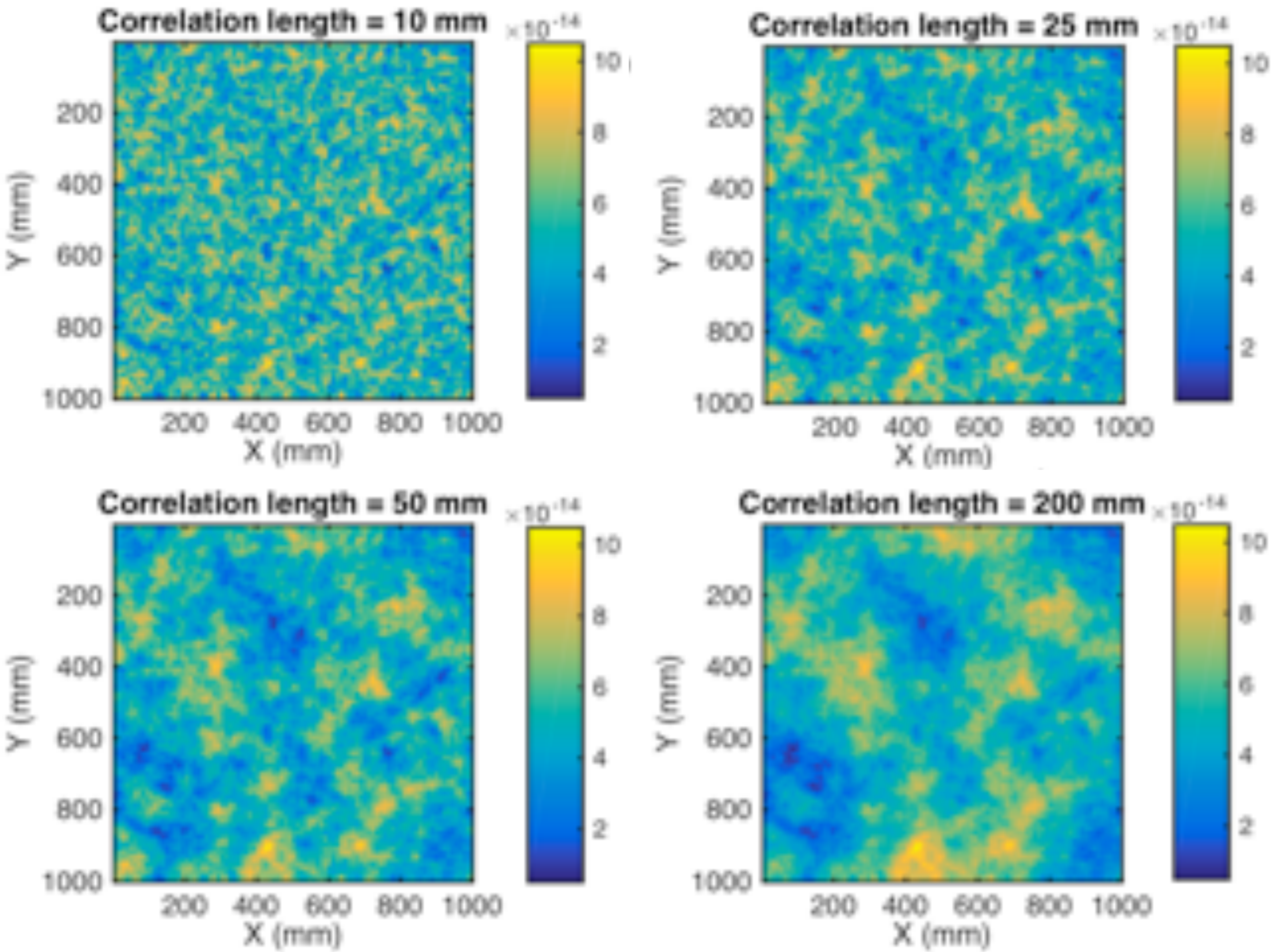
PS74 Part II: Mass Smoothing

Because the minimum mass is set by the minimum spatial scale of the density field $\delta_0(\vec{x})$, one can increase the minimum mass by smoothing the density field to a physical scale of $R = [M/(\gamma\rho_c(z))]^{1/3} > R_{\rm cell}$ and counting the peaks above the threshold $\delta_c(z) = 1.686(1+z)$.



Spatial Smoothing of Density Fluctuation Fields is Convolution

Mathematically: $(\delta * w_R)(\vec{x}) = \int \delta(\vec{x}')w_R(\vec{x}-\vec{x}')d^3\vec{x}'$, where $w_R(\vec{x})$ is the window function

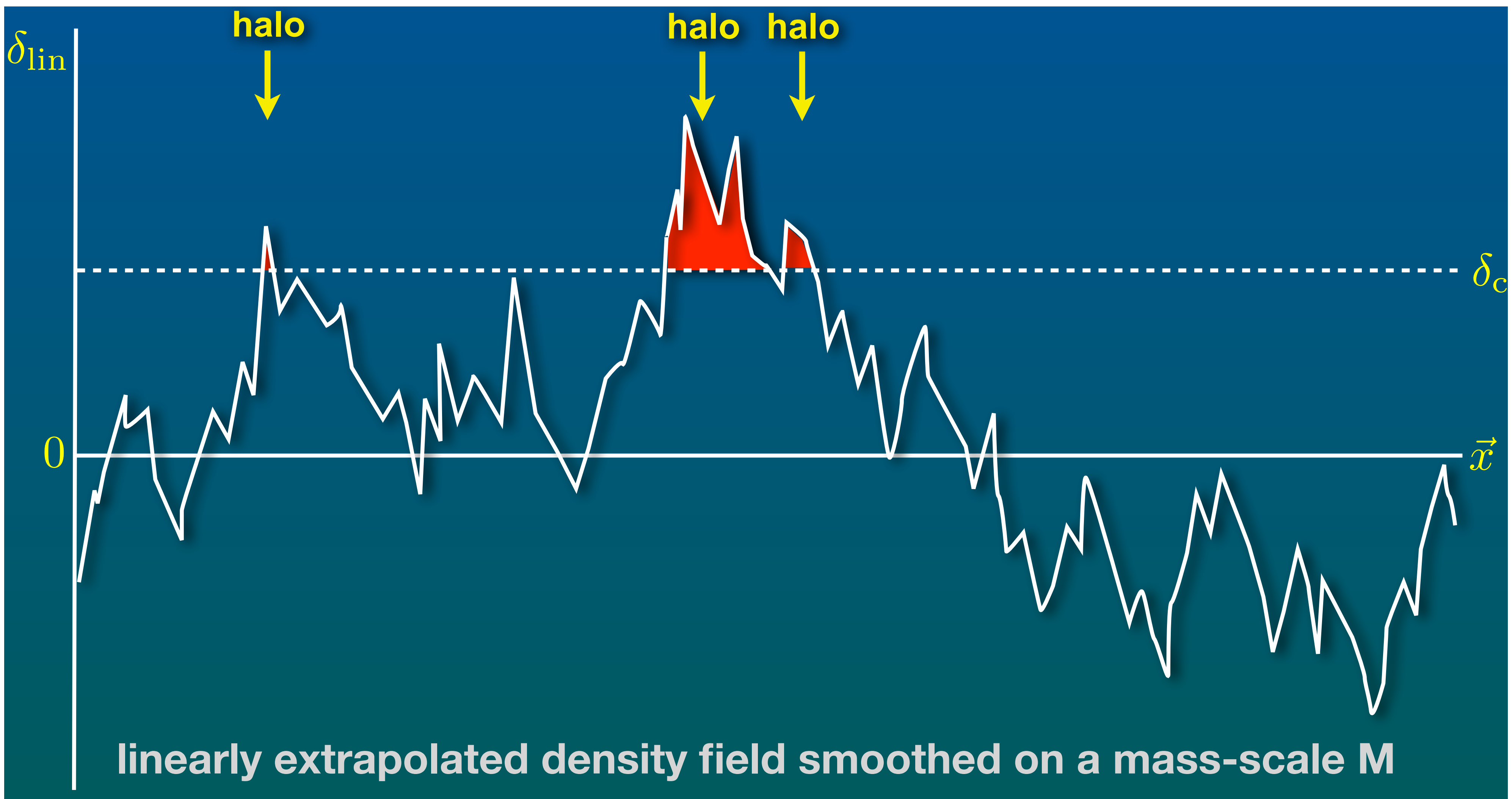


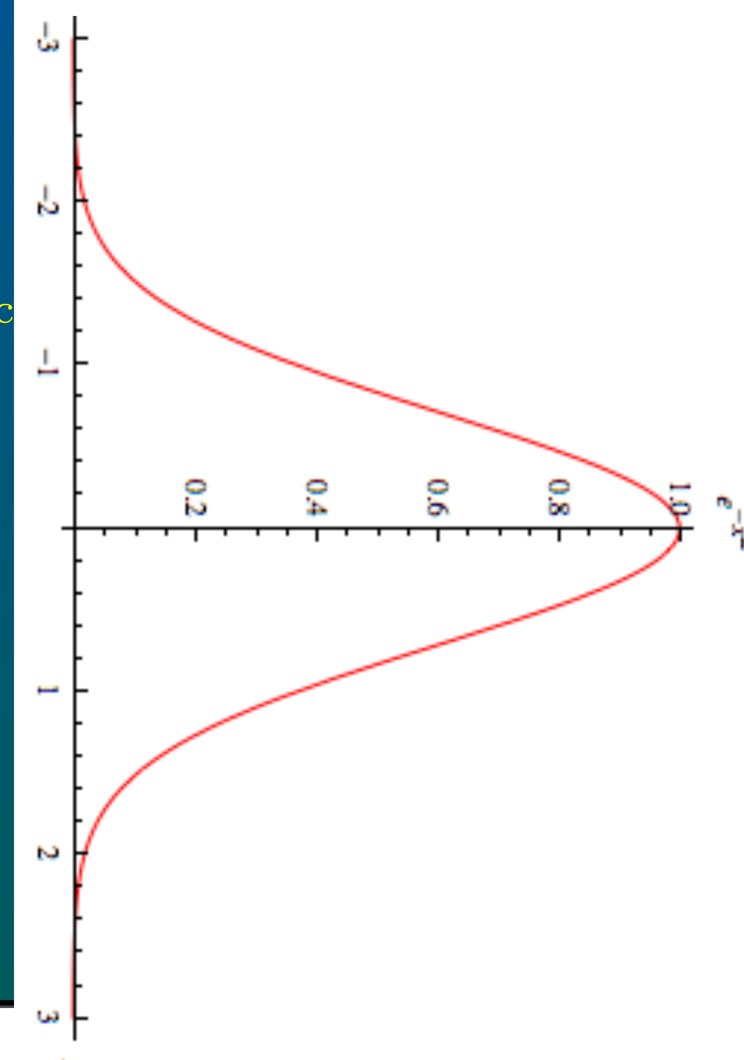
Counting Peaks in the Mass-Smoothed Density Field

The physical scale *R* used to smooth the linearly extrapolated density field $\delta_0(x)$ today determines the minimum mass of all collapsable regions with $\delta_M > \delta_c(z)$, so the fraction of mass locked in halos with masses greater than $M \equiv \gamma\rho_c R^3$ is

$$F(>M) = P[\delta_M > \delta_c(z)];\ \text{where}\ \delta_M(x) = \int \delta_0(x')w[x-x';(M/\gamma\rho_c)^{1/3}]d^3x'$$

where w(x; R) is the window function used for smoothing





Calculate the probability above the collapse threshold

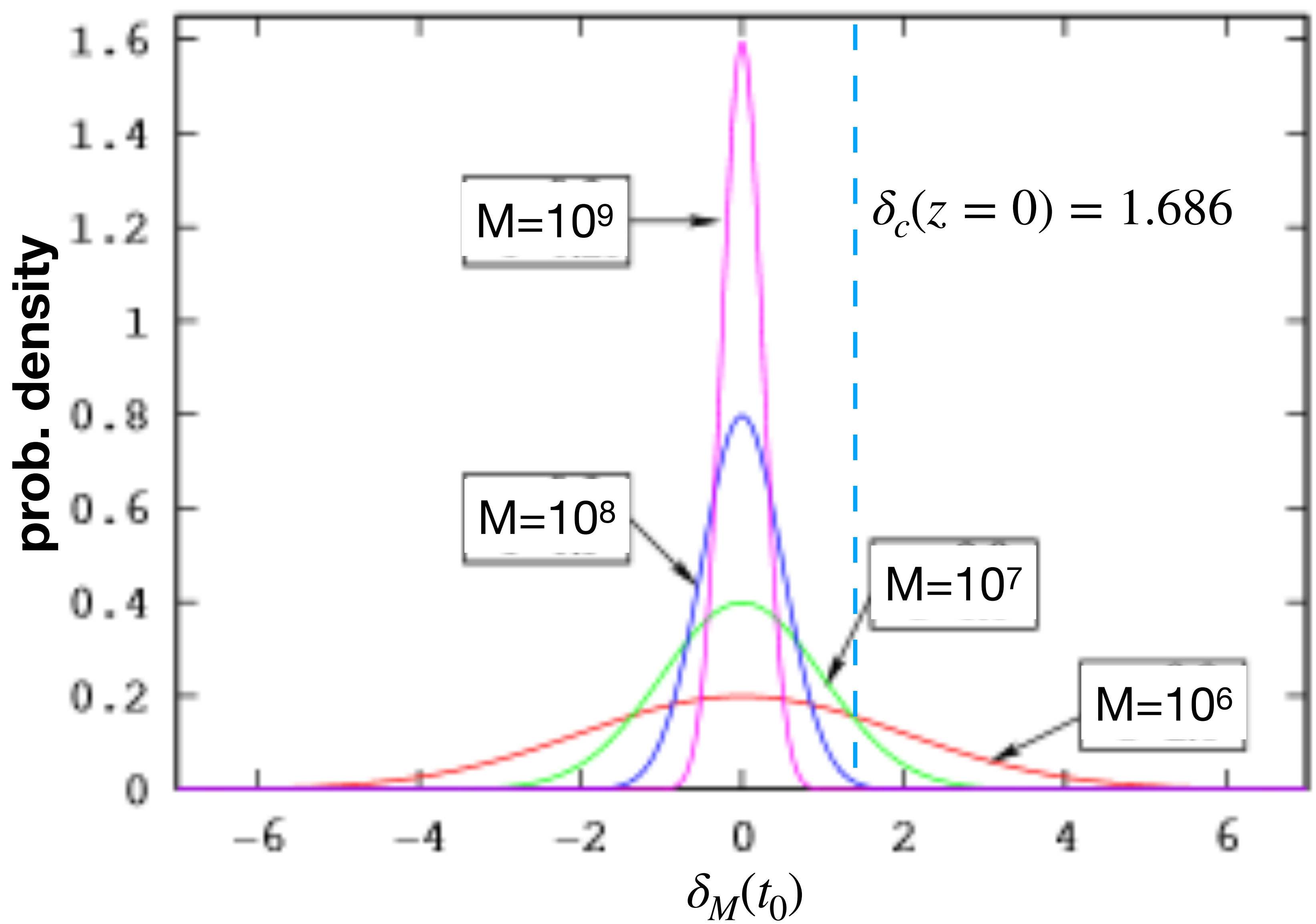
For **Gaussian** random fields, the prob. of finding an overdensity greater than a threshold is:

$$F(>M) = P(\delta_M > \delta_c) = \int_{\delta_c}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_M} \exp\left(-\frac{\delta_M^2}{2\sigma_M^2}\right) d\delta_M = \frac{1}{2}\mathrm{erfc}\left(\frac{\delta_c}{\sqrt{2}\sigma_M}\right)$$

the result only depends on:

(1) the threshold $\delta_c(z) = 1.686(1+z)$, and

(2) the variance of the smoothed field σ_M, which **decreases** as ***M*** increases!



Convolution of the density fluctuation field in spatial dimension decreases σ_M (the width of the Gaussian distribution); i.e., higher mass halos are less likely to form than lower mass halos

PS74 Part III: Cumulative to Differential Distribution

From *Cumulative* mass-in-halo fraction to *Differential* Halo Mass Function

The PS postulate: the fraction of mass locked up in halos w/ mass > M is (*fudge factor 2 is used to account for over-density:* $\rho/\bar{\rho} = 1 + \delta \approx 2$):

$$F(>M, z) = 2P[\delta_M > \delta_c(z)] = \mathrm{erfc}\left(\frac{\delta_c}{\sqrt{2}\sigma_M}\right)$$

the fraction of mass locked up in halos in the mass range [M,M+dM] is:

$$\frac{dF(>M)}{dM}dM = 2\frac{dP}{dM}dM = 2\frac{dP}{d\sigma_M}\frac{d\sigma_M}{dM}dM$$

multiplying the above differential mass fraction by the cosmic mean density $\bar{\rho} = \rho_c$ gives the total collapsed mass per unit volume, which is then divided by M to give the volume density of halos with masses between [M, M+dM], i.e., $\phi(M, z)dM$:

$$\phi dM = \frac{\bar{\rho}}{M}\frac{dF(>M, z)}{dM}dM = 2\frac{\bar{\rho}}{M}\frac{dP}{d\sigma_M}\frac{d\sigma_M}{dM}dM;$$

$$\frac{dP}{d\sigma_M} = \frac{1}{\sqrt{2\pi}}\frac{\delta_c}{\sigma_M^2}\exp\left(-\frac{\delta_c^2}{2\sigma_M^2}\right);\ \text{also}\ \frac{d\sigma_M}{dM} = \frac{\sigma_M}{M}\frac{d\ln\sigma_M}{d\ln M}$$

we have the **final result**:

$$\phi(M, z) = \sqrt{\frac{2}{\pi}}\frac{\bar{\rho}}{M^2}\frac{\delta_c}{\sigma_M}\exp\left(-\frac{\delta_c^2}{2\sigma_M^2}\right)\left|\frac{d\ln\sigma_M}{d\ln M}\right|$$

PS74 Part IV: Comparison with Schechter Luminosity Function

$$\phi(M,z) = \sqrt{\frac{2}{\pi}} \frac{\bar{\rho}}{M^2} \frac{\delta_c}{\sigma_M} \left| \frac{d \ln \sigma_M}{d \ln M} \right| \exp\left(-\frac{\delta_c^2}{2\sigma_M^2} \right) \quad \text{-->} \quad \phi(M) = \frac{\mathrm{d}N}{\mathrm{d}V\mathrm{d}M} = \frac{\phi_\star}{M_\star} \left(\frac{M}{M_\star} \right)^{-1.5} \exp\left(-\frac{M}{M_\star} \right)$$

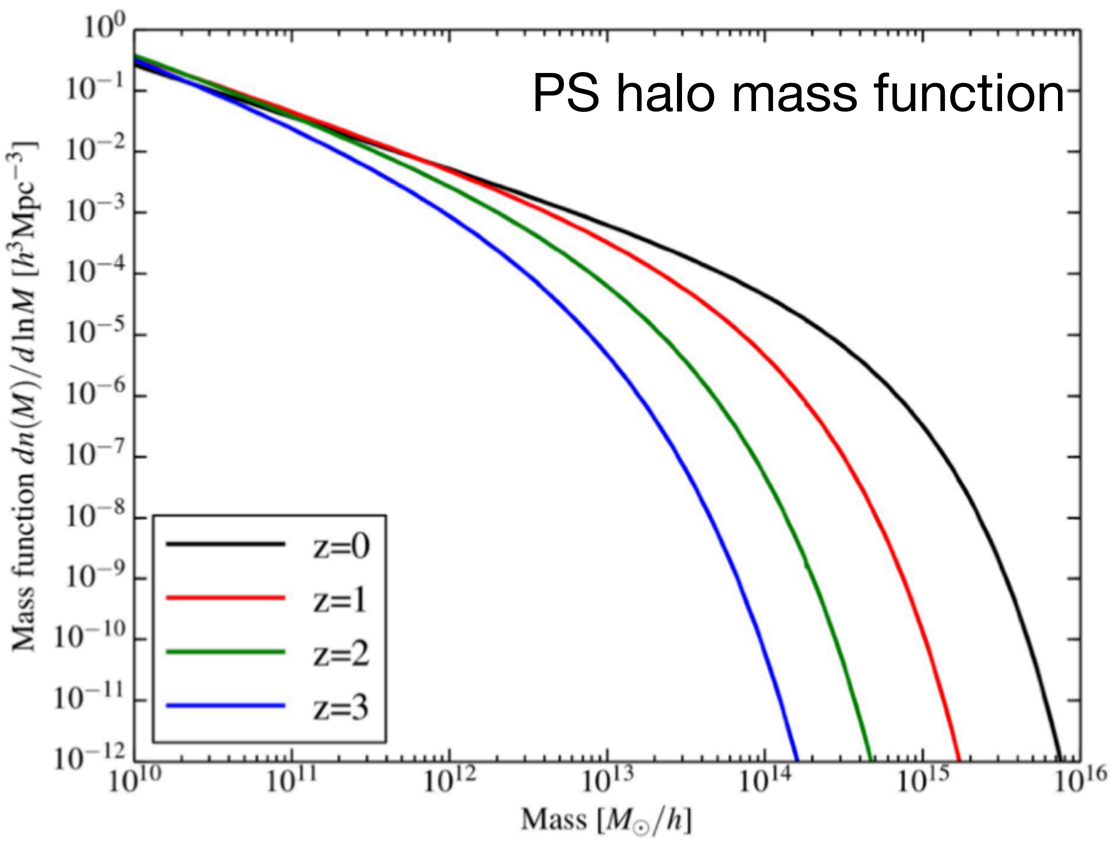
If we define a characteristic mass at redshift z, $M_\star(z)$, by requiring

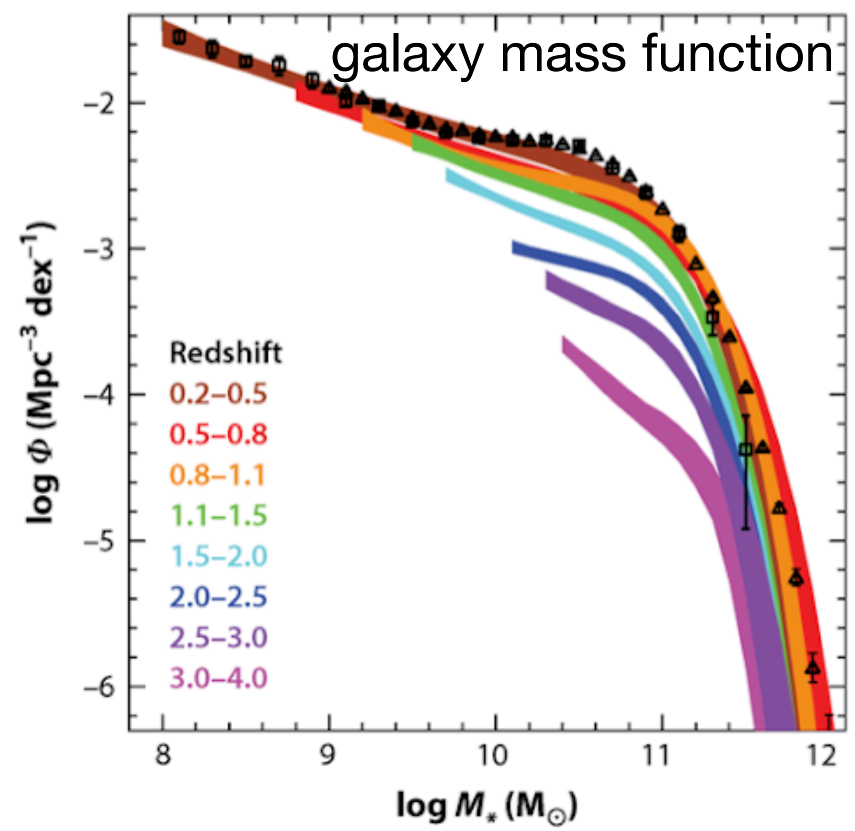
$$2\sigma_M^2[M_\star(z)] = \delta_c^2(z) = [1.686(1+z)]^2,$$

we'll see the behavior of the predicted mass function at the two extremes:

- When $M \ll M_\star$, $\sigma_M \gg \delta_c$, the exponential component approaches unity. If $\sigma_M \propto M^{-\alpha}$, then $\sigma_M = \delta_c (M/M_\star)^{-\alpha}/\sqrt{2}$, and $\phi(M) \propto (M/M_\star)^{\alpha-2}$
- When $M \gg M_\star$, $\sigma_M \ll \delta_c$, the exponential component becomes important. If $\sigma_M \propto M^{-\beta}$, then $2\sigma_M^2 = \delta_c^2 (M/M_\star)^{-\beta}$, and $\phi(M) \propto \exp[-(M/M_\star)^\beta]$

Both behaviors are similar to the Schechter function for the observed distribution functions, and a forced match would lead to $\alpha \approx 0.5, \beta = 1$





Recap of previous lecture

Derivation of PS Halo Mass Function

The PS postulate: the fraction of mass locked up in halos w/ mass > M is (*fudge factor 2 is used to account for over-density:* $\rho/\bar{\rho} = 1 + \delta \approx 2$):

$$F(>M, z) = 2P[\delta_M > \delta_c(z)] = \mathrm{erfc}\left(\frac{\delta_c}{\sqrt{2}\sigma_M}\right)$$

the fraction of mass locked up in halos in the mass range [M,M+dM] is:

$$\frac{dF(>M)}{dM}dM = 2\frac{dP}{dM}dM = 2\frac{dP}{d\sigma_M}\frac{d\sigma_M}{dM}dM$$

multiplying the above differential mass fraction by the cosmic mean density $\bar{\rho} = \rho_c$ gives the total collapsed mass per unit volume, which is then divided by M to give the volume density of halos with masses between [M, M+dM], i.e., $\phi(M, z)dM$:

$$\phi dM = \frac{\bar{\rho}}{M}\frac{dF(>M, z)}{dM}dM = 2\frac{\bar{\rho}}{M}\frac{dP}{d\sigma_M}\frac{d\sigma_M}{dM}dM;$$

$$\frac{dP}{d\sigma_M} = \frac{1}{\sqrt{2\pi}}\frac{\delta_c}{\sigma_M^2}\exp(-\frac{\delta_c^2}{2\sigma_M^2});\ \text{also}\ \frac{d\sigma_M}{dM} = \frac{\sigma_M}{M}\frac{d\ln\sigma_M}{d\ln M}$$

we have the **final result**:

$$\phi(M, z) = \sqrt{\frac{2}{\pi}}\frac{\bar{\rho}}{M^2}\frac{\delta_c}{\sigma_M}\left|\frac{d\ln\sigma_M}{d\ln M}\right|\exp\left(-\frac{\delta_c^2}{2\sigma_M^2}\right)$$

$$\phi(M,z) = \sqrt{\frac{2}{\pi}} \frac{\bar{\rho}}{M^2} \frac{\delta_c}{\sigma_M} \left| \frac{d \ln \sigma_M}{d \ln M} \right| \exp\left(-\frac{\delta_c^2}{2\sigma_M^2} \right) \quad \text{-->} \quad \phi(M) = \frac{\mathrm{d}N}{\mathrm{d}V\mathrm{d}M} = \frac{\phi_\star}{M_\star} \left(\frac{M}{M_\star} \right)^{-1.5} \exp\left(-\frac{M}{M_\star} \right)$$

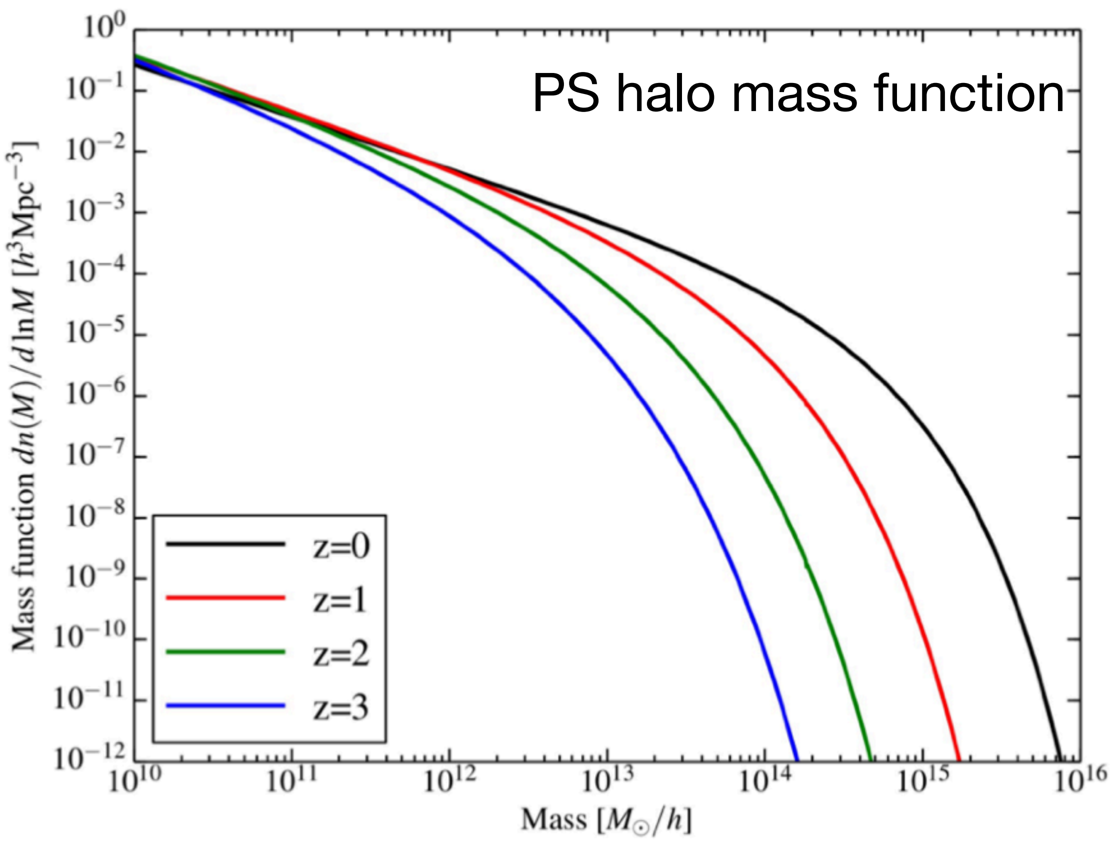
If we define a characteristic mass at redshift z, $M_\star(z)$, by requiring

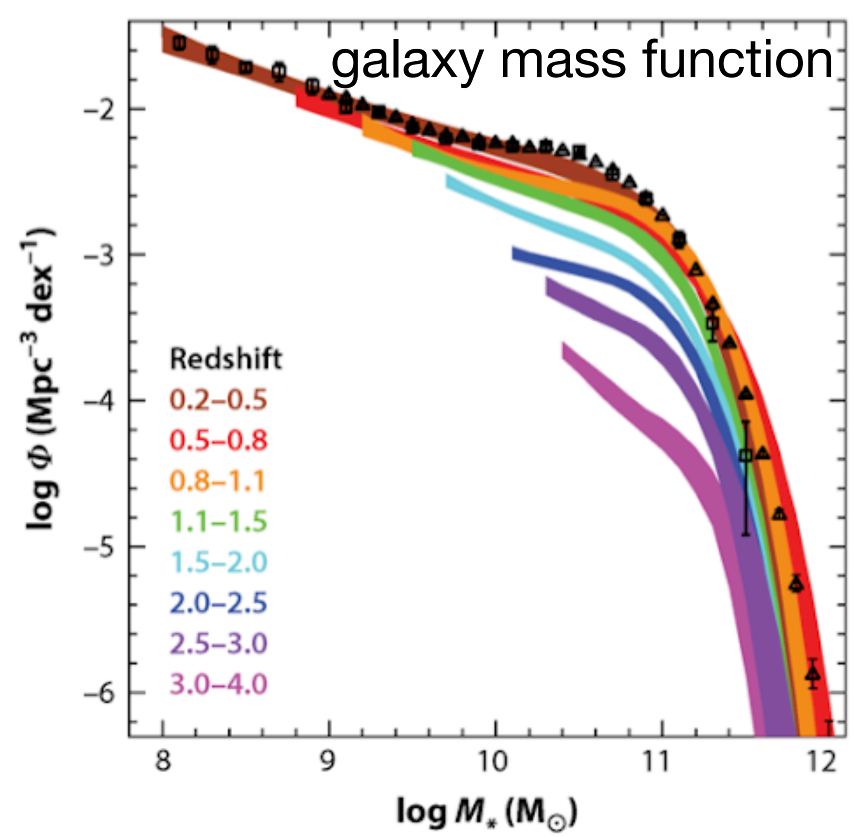
$$2\sigma_M^2[M_\star(z)] = \delta_c^2(z) = [1.686(1+z)]^2,$$

we'll see the behavior of the predicted mass function at the two extremes:

- When $M \ll M_\star$, $\sigma_M \gg \delta_c$, the exponential component approaches unity. If $\sigma_M \propto M^{-\alpha}$, then $\sigma_M = \delta_c (M/M_\star)^{-\alpha}/\sqrt{2}$, and $\phi(M) \propto (M/M_\star)^{\alpha-2}$
- When $M \gg M_\star$, $\sigma_M \ll \delta_c$, the exponential component becomes important. If $\sigma_M \propto M^{-\beta}$, then $2\sigma_M^2 = \delta_c^2 (M/M_\star)^{-\beta}$, and $\phi(M) \propto \exp[-(M/M_\star)^\beta]$

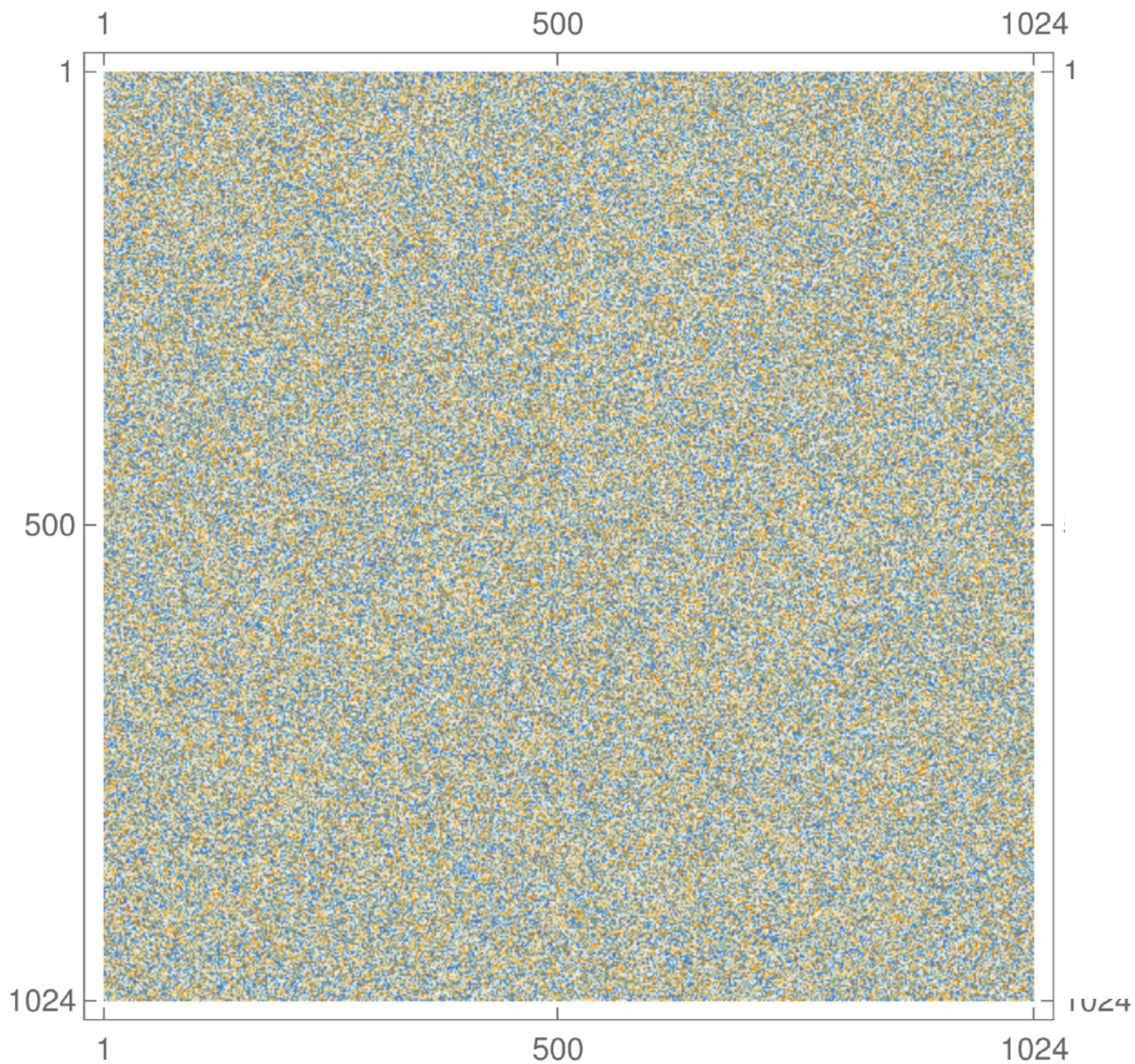
Both behaviors are similar to the Schechter function for the observed distribution functions, and a forced match would lead to $\alpha \approx 0.5, \beta = 1$



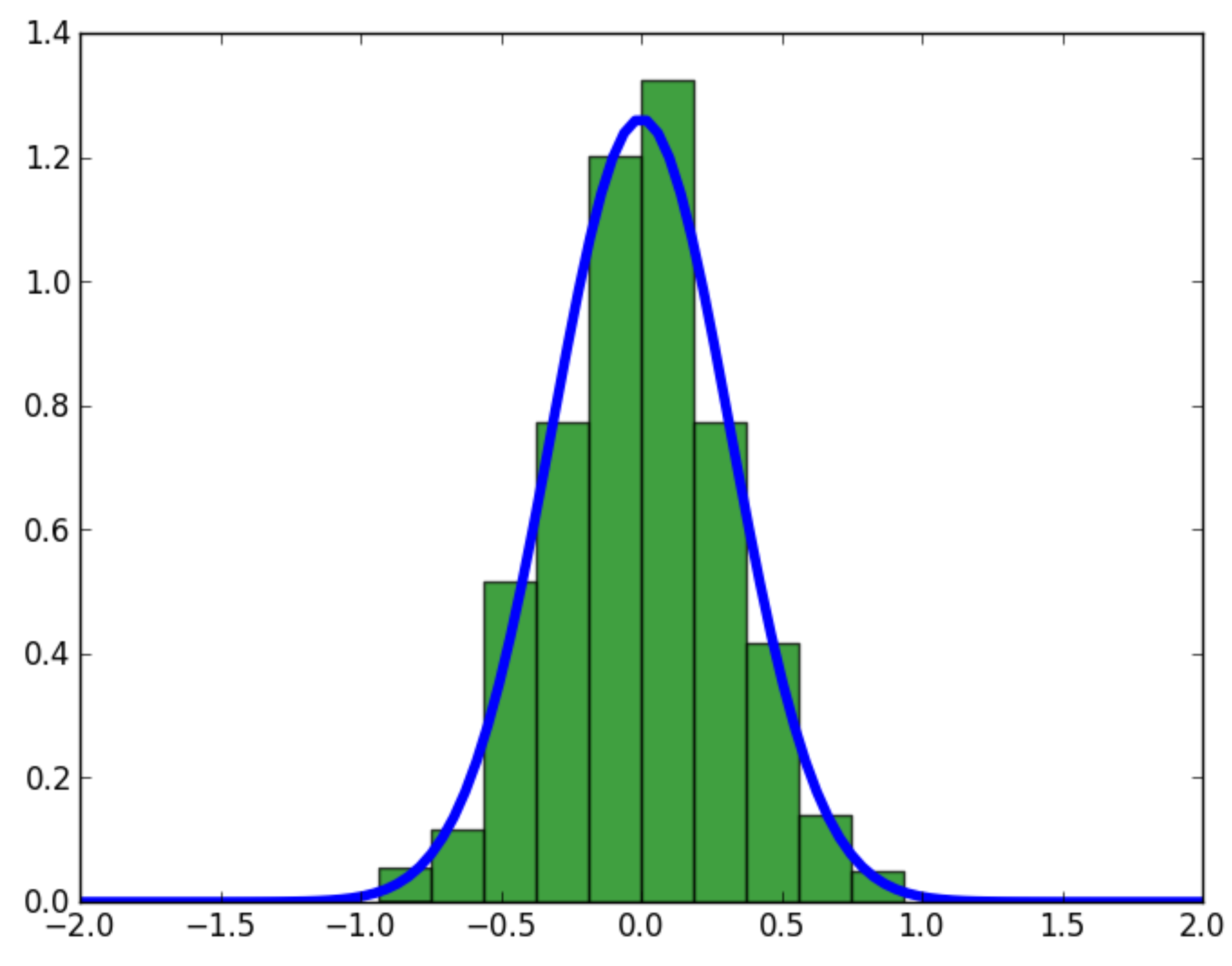


independent/uncorrelated Gaussian fluctuations
White Noise

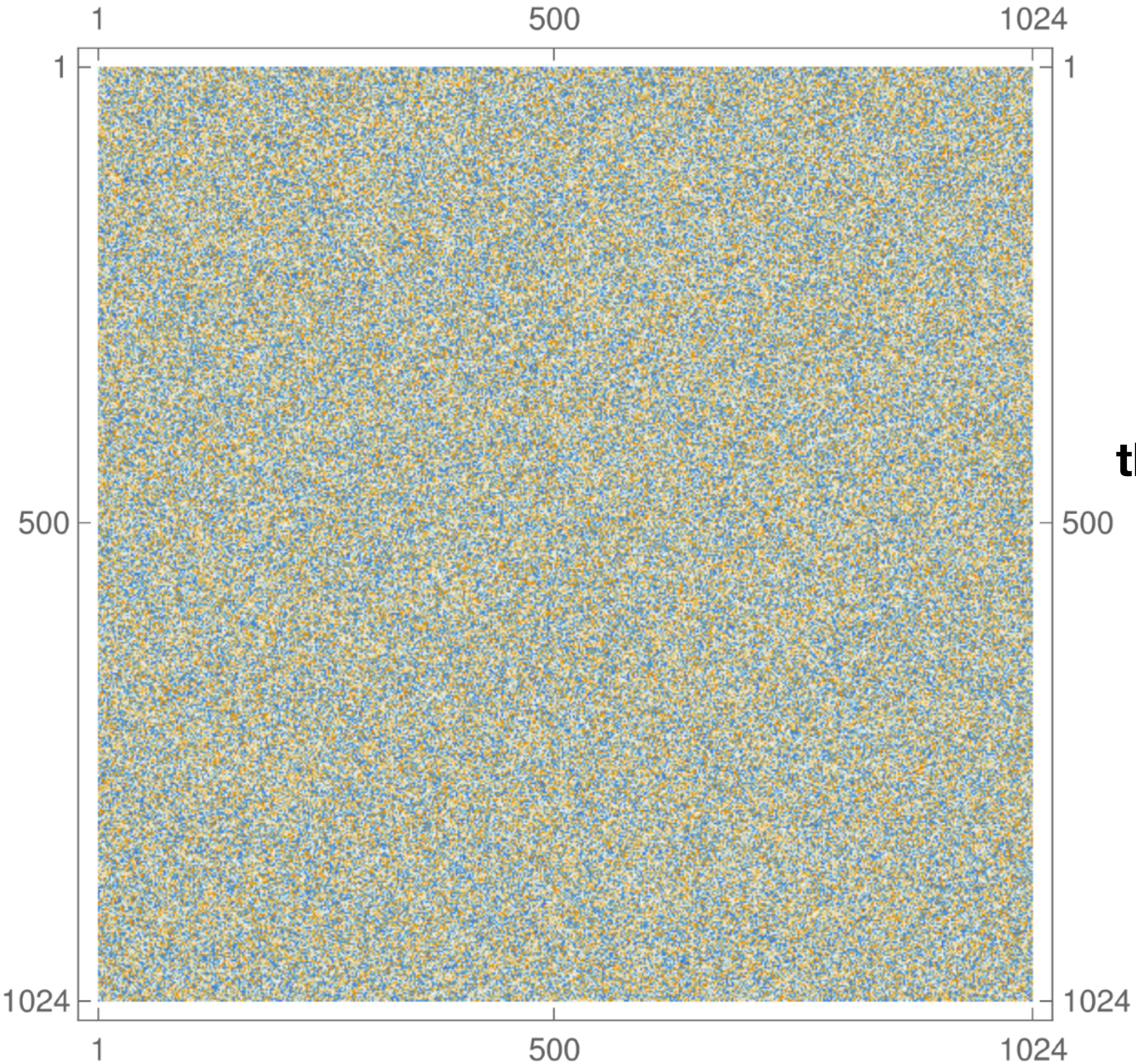
Img = randomn(seed, nx=1024, ny=1024, /normal, **sigma**=0.2)



histogram(img, bin=0.2)



Mass variance expected from White Noise



Spatial smoothing will reduce the variance of the white noise as:

$$\sigma_M^2 \propto 1/N_M$$

where N_M is the number of pixels that went into the smoothing kernel:

$$N \propto R^3 \propto M$$

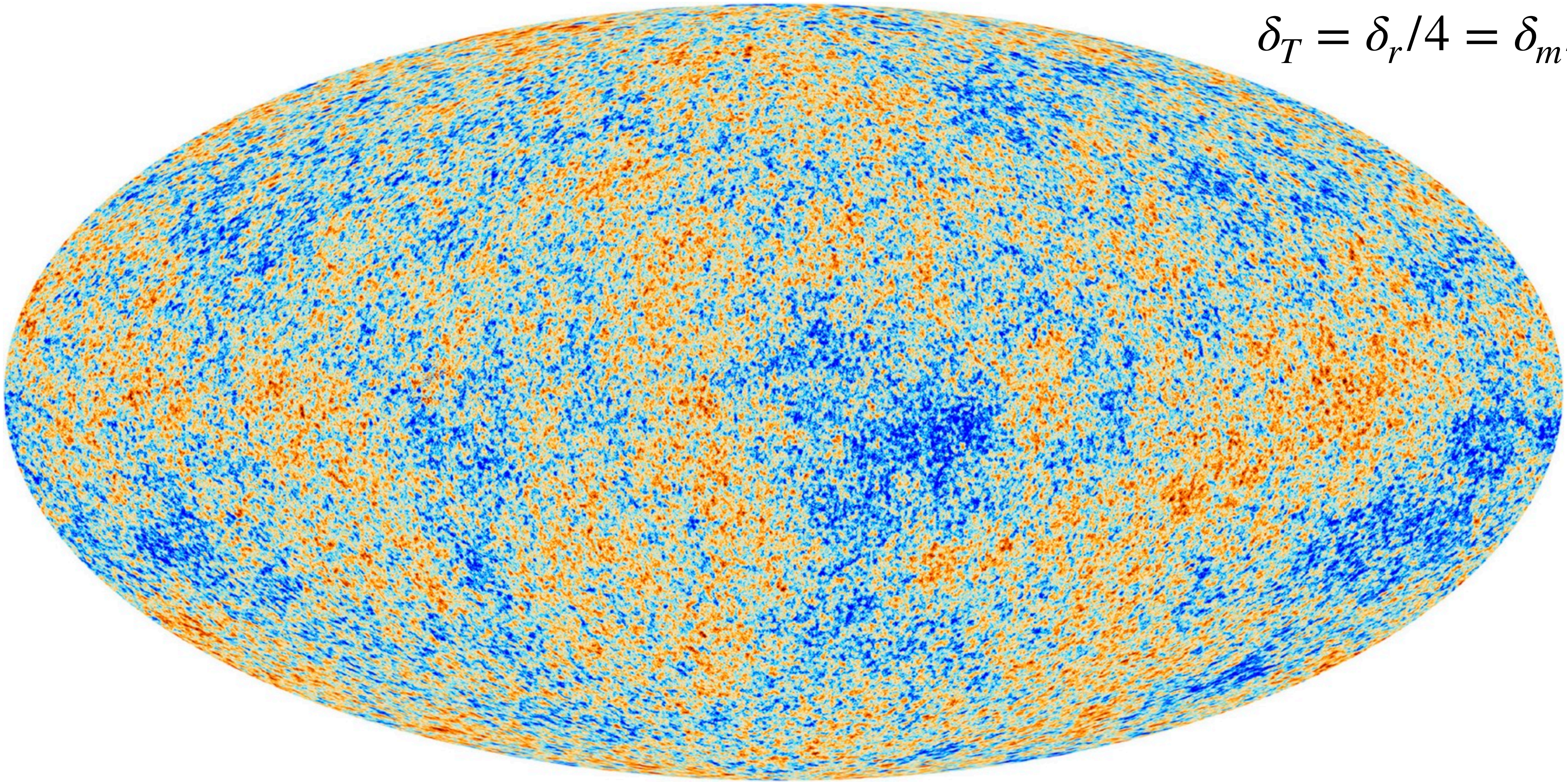
as a result, one expects:

$$\sigma_M \propto M^{-1/2}$$

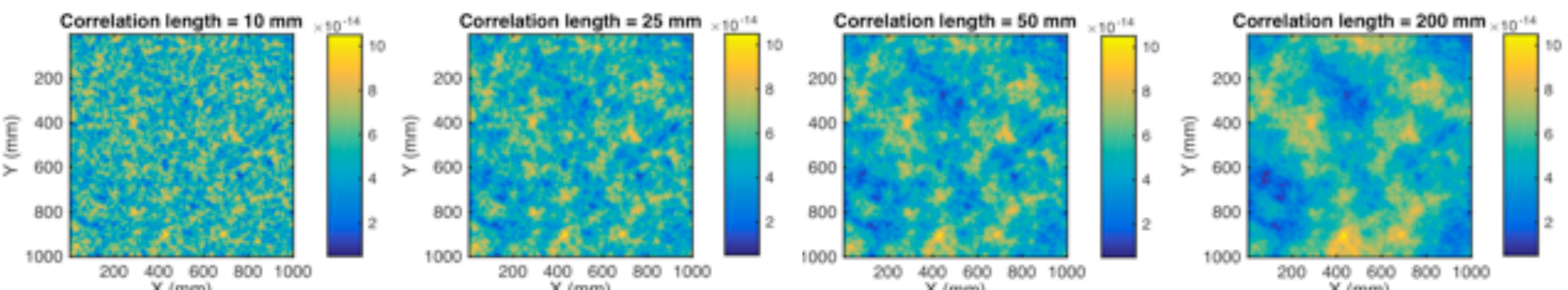
Planck CMB δ_T map is not uncorrelated Gaussian noise map

2018 Planck Map of Temperature Fluctuations

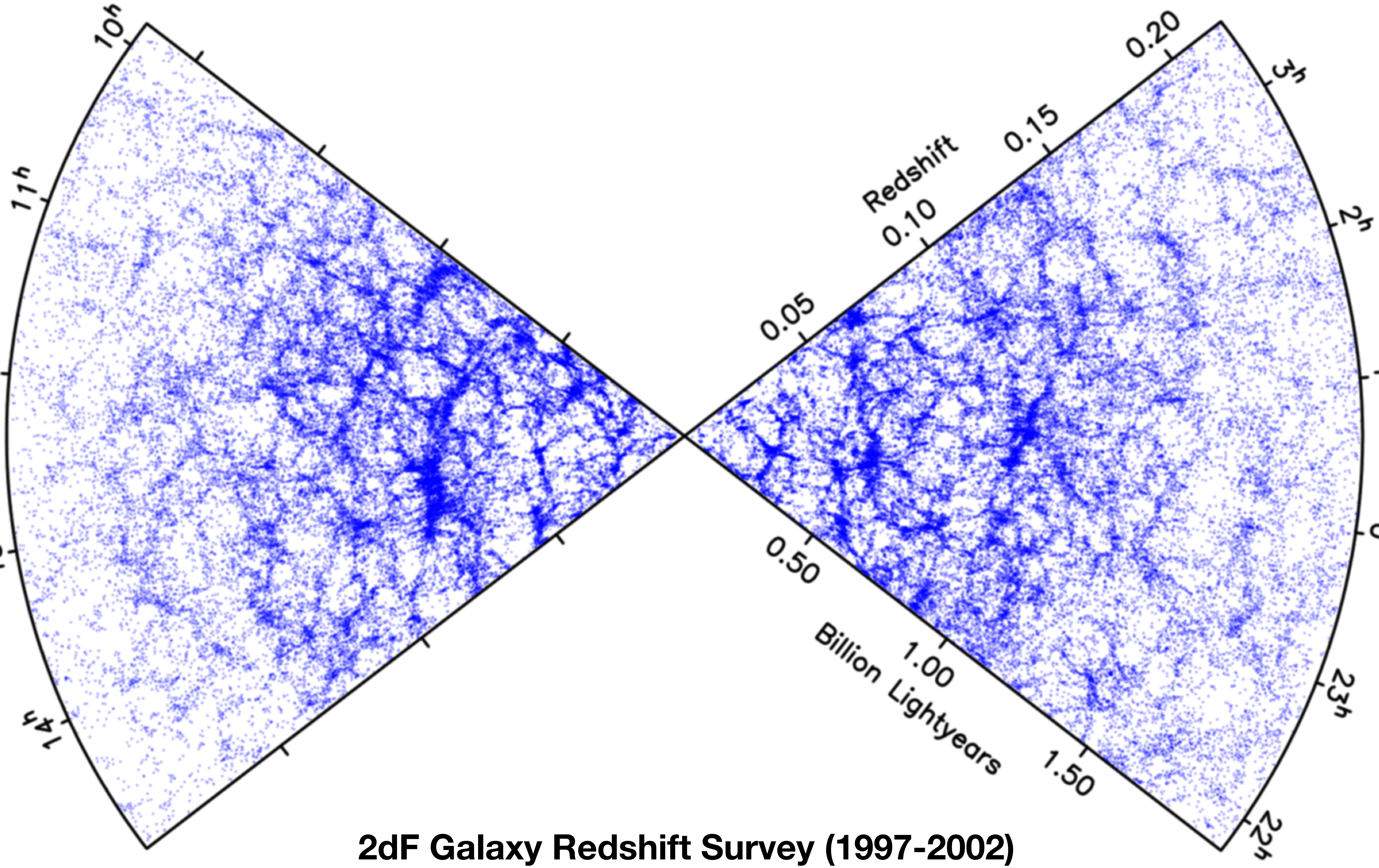
$$\delta_T = \delta_r/4 = \delta_m/3$$



Gaussian random fields w/ increasing correlation lengths



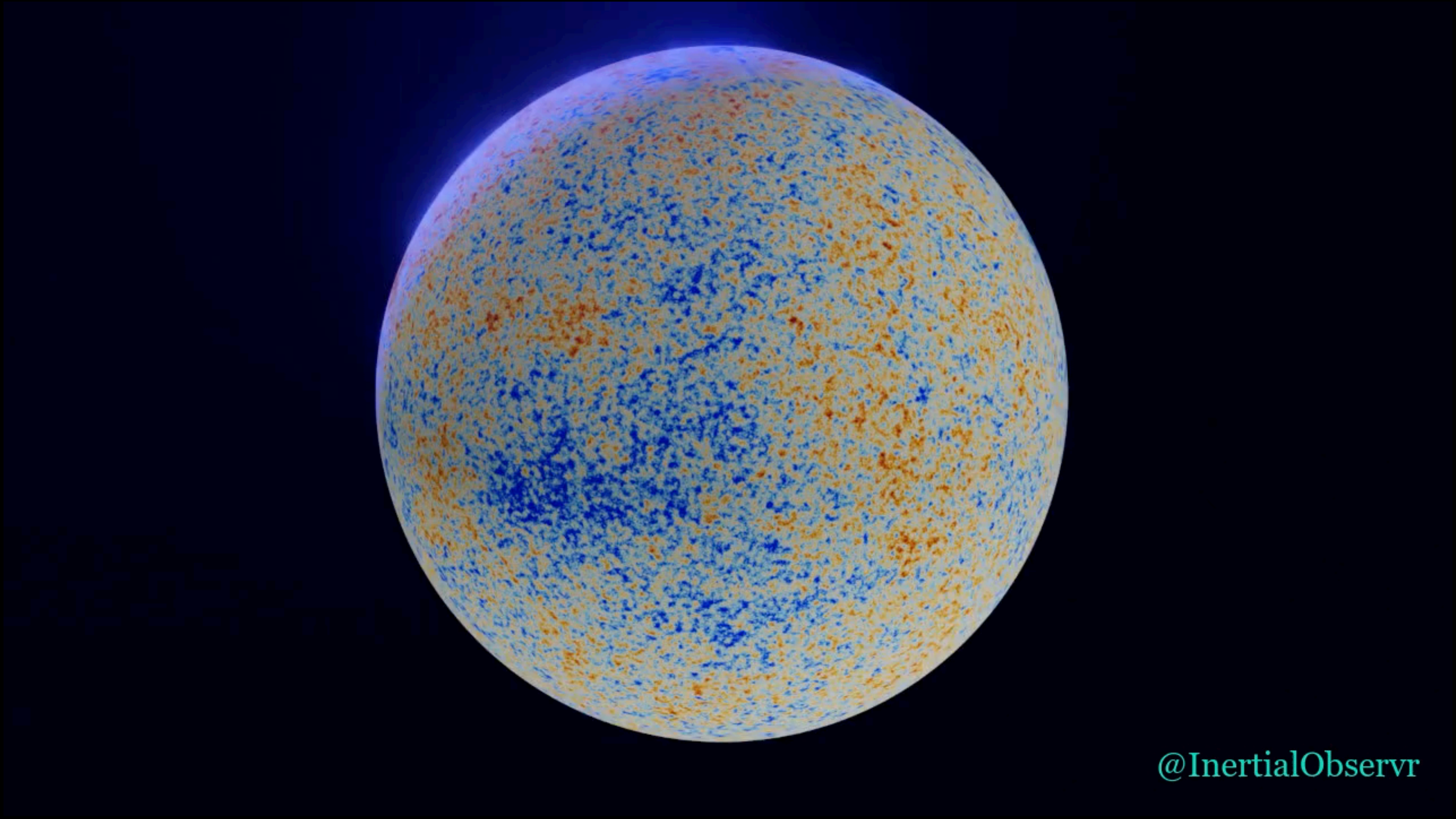
Galaxy distribution also doesn't look like white noise either



2dF Galaxy Redshift Survey (1997-2002)

A Gaussian random fluctuation field is determined by:

1. the amplitude of the fluctuation (σ), and
2. the spatial correlation between fluctuations



calculation of σ_M

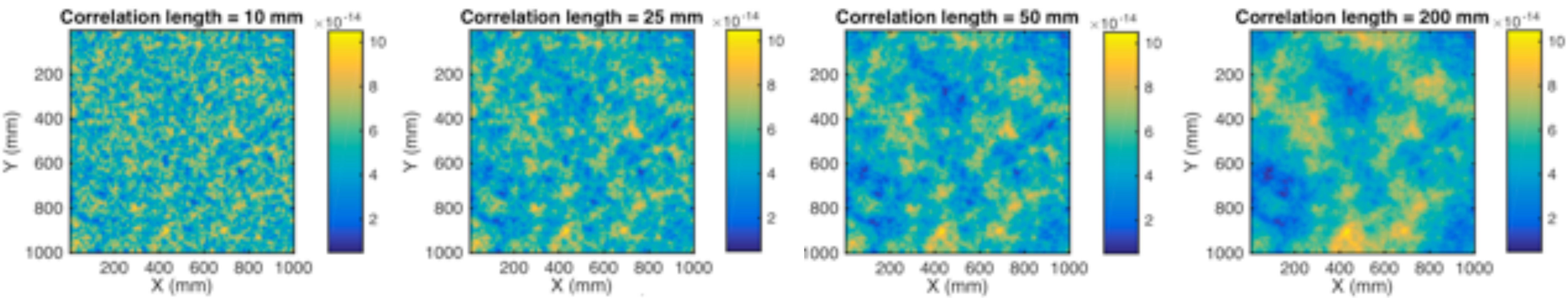
smoothing / convolution in real space

$$\sigma_M^2 = \frac{1}{V}\int_{-\infty}^{\infty} \delta_M^2(x) d^3x = \frac{1}{V}\int_{-\infty}^{\infty} |(\delta * w_M)(x)|^2 d^3x$$

Moving beyond white noise …

smoothing is a **convolution** of the density field w/ a window function of width *R* ($M = \gamma\bar{\rho}R^3$):

$$\delta_M(x) = (\delta * w_M)(x) = \int \delta(x')w_M(x' - x)d^3x'$$



the **variance** of the **smoothed** density field is then:

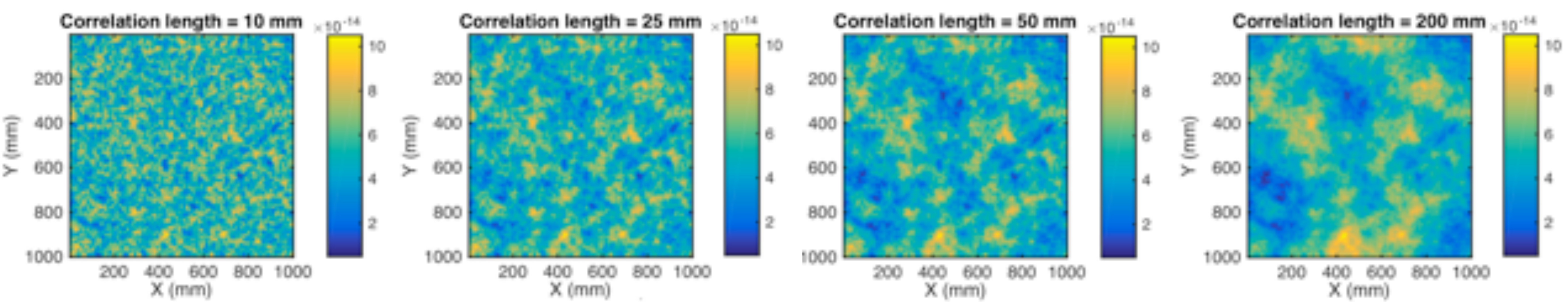
$$\sigma_M^2 = \langle \delta_M^2(x) \rangle = \frac{1}{V}\int \delta_M^2(x)d^3x = \frac{1}{V}\int |(\delta * w_M)(x)|^2 d^3x$$

$$\sigma_M^2 = \frac{1}{V}\int_{-\infty}^{\infty} \delta_M^2(x) d^3x = \frac{1}{V}\int_{-\infty}^{\infty} |(\delta * w_M)(x)|^2 d^3x$$

There are some major problems going forward:

- *what would be the appropriate volume and what scale to use to sample this random field?*
- *numerical convolution is computationally expensive and how to deal with artifacts near boundaries?*

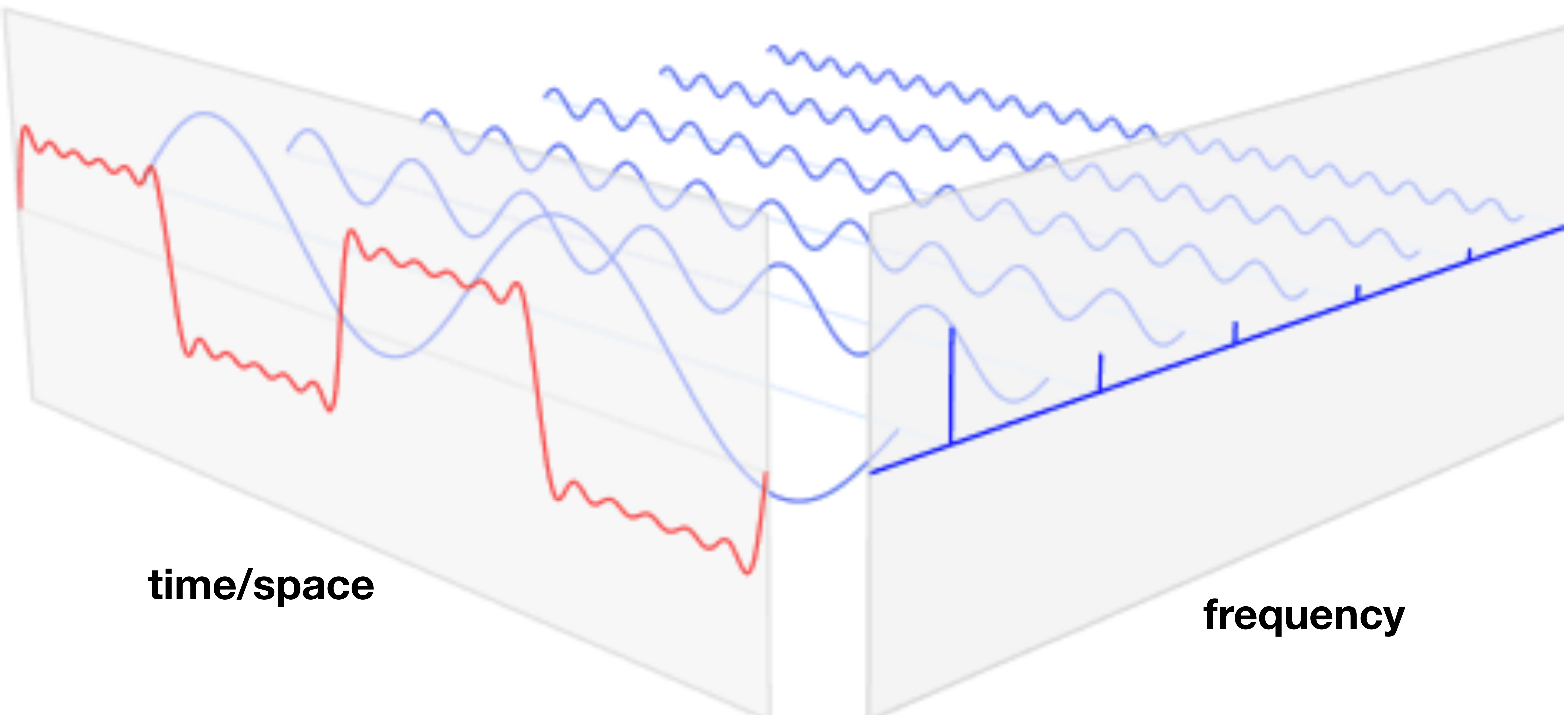
- *The density field is a random field, so would require many realizations,*
- *how to realize random fields that match the initial conditions of the universe?*



Fourier Transform:

1. Convolution Theorem
2. Parseval’s Identity

Consider transforming to Fourier space



Decomposing a *periodic* time/space signal into Fourier series

The Fourier transform

we'll be interested in signals defined for all t

the **Fourier transform** of a signal f is the function

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

- F is a function of a *real* variable ω; the function value $F(\omega)$ is (in general) a complex number

$$F(\omega) = \int_{-\infty}^{\infty} f(t) \cos \omega t \, dt - j \int_{-\infty}^{\infty} f(t) \sin \omega t \, dt$$

- $|F(\omega)|$ is called the *amplitude spectrum* of f; $\angle F(\omega)$ is the *phase spectrum* of f
- notation: $F = \mathcal{F}(f)$ means F is the Fourier transform of f; as for

We adopt the non-unitary definition w/ angular frequencies

Summary of popular forms of the Fourier transform, one-dimensional

ordinary frequency ξ (Hz)	**unitary**	$\widehat{f_1}(\xi) \triangleq \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi\xi x}\, dx = \sqrt{2\pi}\ \widehat{f_2}(2\pi\xi) = \widehat{f_3}(2\pi\xi)$ $f(x) = \int_{-\infty}^{\infty} \widehat{f_1}(\xi)\, e^{i2\pi x\xi}\, d\xi$
angular frequency ω (rad/s)	**unitary**	$\widehat{f_2}(\omega) \triangleq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx = \frac{1}{\sqrt{2\pi}}\ \widehat{f_1}\left(\frac{\omega}{2\pi}\right) = \frac{1}{\sqrt{2\pi}}\ \widehat{f_3}(\omega)$ $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f_2}(\omega)\, e^{i\omega x}\, d\omega$
	non-unitary	$\widehat{f_3}(\omega) \triangleq \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx = \widehat{f_1}\left(\frac{\omega}{2\pi}\right) = \sqrt{2\pi}\ \widehat{f_2}(\omega)$ $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f_3}(\omega)\, e^{i\omega x}\, d\omega$

FT and inverse FT in 1D

time domain:

$$F(\omega) = \int f(t)\ e^{-i\omega t} dt$$

$$f(t) = \frac{1}{2\pi} \int F(\omega) e^{i\omega t} d\omega$$

angular freq.:

$$\omega = \frac{2\pi}{\Delta t}$$

space domain:

$$F(k) = \int f(x)\ e^{-ikx} dx$$

$$f(x) = \frac{1}{2\pi} \int F(k) e^{ikx} dk$$

wave number:

$$k = \frac{2\pi}{\Delta x}$$

F.T. of 3D Density Perturbation Field

1D space domain:

$$f(x) = \frac{1}{2\pi} \int F(k) e^{ikx} dk$$

$$F(k) = \int f(x)\ e^{-ikx} dx$$

wave number:

$$k = \frac{2\pi}{\Delta x}$$

3D space domain:

$$\delta(\vec{x}) = \frac{1}{(2\pi)^3} \int \Delta(\vec{k})\ e^{+i\vec{k}\cdot\vec{x}} d^3\vec{k}$$

$$\Delta(\vec{k}) = \int \delta(\vec{x})\ e^{-i\vec{k}\cdot\vec{x}} d^3\vec{x}$$

Convolution Theorem

convolution in real space = **multiplication** in Fourier space

$$\mathscr{F}\{(f * g)(x)\} = \mathscr{F}\{f\}(k) \cdot \mathscr{F}\{g\}(k) = F(k) \cdot G(k)$$

Application on the mass-filtered density perturbation field:

$$\mathscr{F}\{(\delta * w_M)(x)\} = \Delta(k) W_M(k)$$

mass-filtered density perturbation field can be computed as the inverse FFT of the product of the FT of density field and the FT of window function

$$\Rightarrow (\delta * w_M)(x) = \mathscr{F}^{-1}\{\Delta(k) W_M(k)\}$$

Parseval's identity (aka Rayleigh's energy theorem)

The identity asserts the equality of the energy of a periodic signal (given as the integral of the squared amplitude of the signal) and the energy of its frequency domain representation (given as the sum of squares of the amplitudes).

$$\int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 \, d\xi = \int_{-\infty}^{\infty} |f(x)|^2 \, dx.$$

Application on the **variance** of density perturbation field:

$$\sigma^2 = \frac{1}{V}\int_{-\infty}^{\infty} \delta^2(x) d^3x = \frac{V}{(2\pi)^3}\int_{-\infty}^{\infty} \Delta^2(k) d^3k$$

Mass variance in Fourier space

The variance of mass-filtered density perturbation field is defined as:

$$\sigma_M^2 = \frac{1}{V}\int_{-\infty}^{\infty} \delta_M^2(x)d^3x = \frac{1}{V}\int_{-\infty}^{\infty} |(\delta * W_M)(x)|^2 d^3x$$

Parseval's identity:

$$\sigma_M^2 = \frac{1}{V}\int_{-\infty}^{\infty} \delta_M^2(x)d^3x = \frac{V}{(2\pi)^3}\int_{-\infty}^{\infty} \Delta_M^2(k)d^3k$$

Convolution theorem:

$$\Delta_M(k) = \mathscr{F}\{\delta_M(x)\} = \mathscr{F}\{\delta(x) * w_M(x)\} = \Delta(k)W_M(k)$$

Combining the two, we have:

$$\sigma_M^2 = \frac{V}{(2\pi)^3}\int_0^{\infty} \Delta^2(k)W_M^2(k)d^3k$$

calculation of σ_M in Fourier space

$$\sigma_M^2 = \frac{V}{2\pi^2}\int_0^\infty \Delta^2(k) W_M^2(k) k^2 dk$$

Mass variance in Fourier space

The variance of mass-filtered density perturbation field is defined as:

$$\sigma_M^2 = \frac{1}{V}\int_{-\infty}^{\infty} \delta_M^2(x) d^3x = \frac{1}{V}\int_{-\infty}^{\infty} |(\delta * W_M)(x)|^2 d^3x$$

Parseval's identity:

$$\sigma_M^2 = \frac{1}{V}\int_{-\infty}^{\infty} \delta_M^2(x) d^3x = \frac{V}{(2\pi)^3}\int_{-\infty}^{\infty} \Delta_M^2(k) d^3k$$

Convolution theorem:

$$\Delta_M(k) = \mathscr{F}\{\delta_M(x)\} = \mathscr{F}\{\delta(x) * w_M(x)\} = \Delta(k) W_M(k)$$

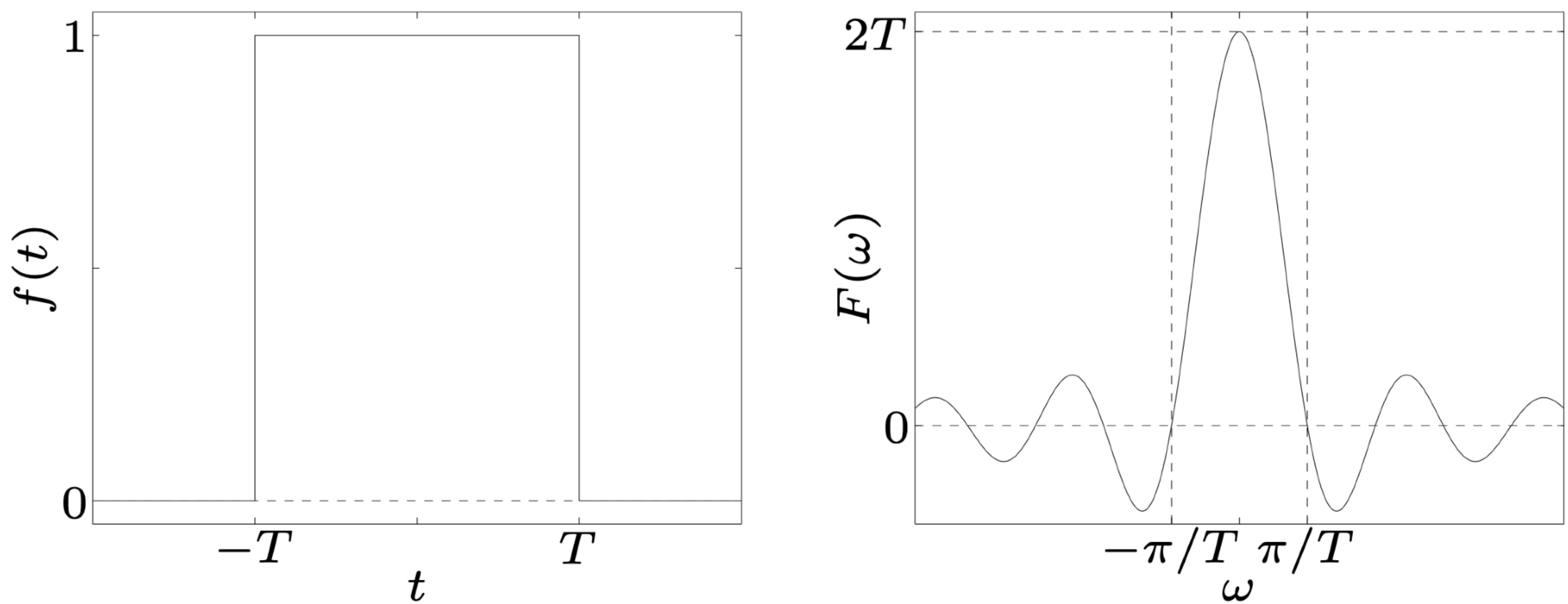
Combining the two, we have:

$$\sigma_M^2 = \frac{V}{(2\pi)^3}\int_{0}^{\infty} \Delta^2(k) W_M^2(k) d^3k$$

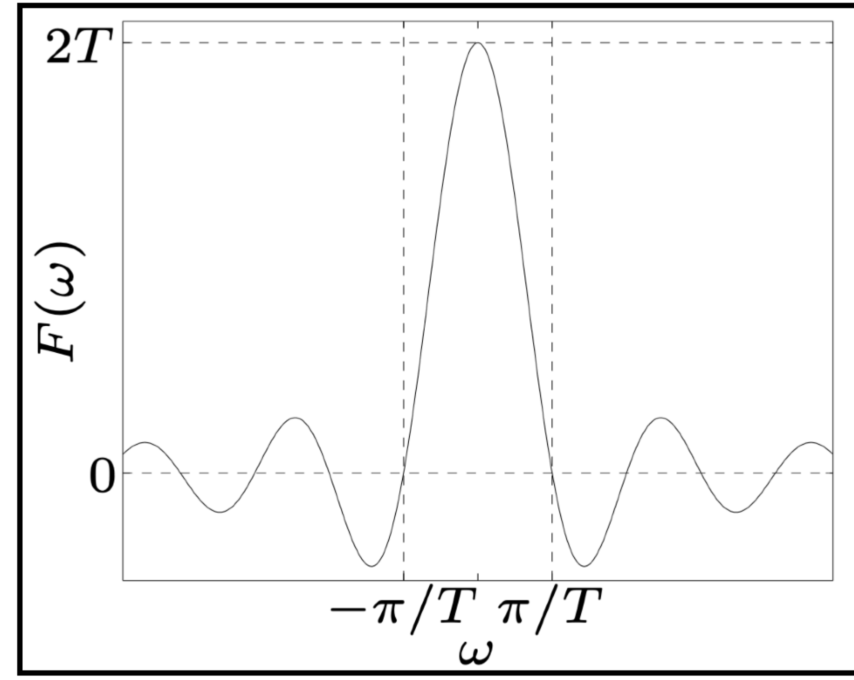
F.T. of Window Function: 1D top-hat

rectangular pulse: $f(t) = \begin{cases} 1 & -T \leq t \leq T \\ 0 & |t| > T \end{cases}$

$$F(\omega) = \int_{-T}^{T} e^{-j\omega t}\, dt = \frac{-1}{j\omega}\left(e^{-j\omega T} - e^{j\omega T}\right) = \frac{2\sin\omega T}{\omega}$$



F.T. of 3D top-hat



rectangular pulse: $f(t) = \begin{cases} 1 & -T \le t \le T \\ 0 & |t| > T \end{cases}$

$$F(\omega) = \int_{-T}^{T} e^{-j\omega t}\, dt = \frac{-1}{j\omega}\left(e^{-j\omega T} - e^{j\omega T}\right) = \frac{2\sin\omega T}{\omega}$$

For a 3D **top hat** window function, we have:

$$w_M(r) = \frac{3}{4\pi R^3} \text{ when } r \le R \;;\; w_M(r) = 0 \text{ when } r > R$$

$$W_M(k) = \frac{3}{(kR)^3}[\sin(kR) - (kR)\cos(kR)]$$

$$\text{where } R = \left(\frac{3M}{4\pi\bar{\rho}}\right)^{1/3} \text{ and } k = \frac{2\pi}{R}$$

Definition of Power Spectrum

Given the previous result from **convolution theorem** and **Parseval's identity**:

$$\sigma_M^2 = \frac{V}{(2\pi)^3} \int_0^\infty \Delta^2(k) W_M^2(k) d^3k$$

If we define **power spectrum:**

$$P(k) = V\Delta^2(k)$$

We obtain:

$$\sigma_M^2 = \frac{1}{2\pi^2} \int_0^\infty P(k) W_M^2(k) k^2 dk$$

Fourier transform of power spectrum

$$P(\vec{k}) = V\Delta^2(\vec{k})$$

define its Fourier Transform as $\xi(x)$

$$P(\vec{k}) = \int \xi(r)\ e^{-i\vec{k}\cdot\vec{r}} d^3\vec{r}$$

unit: Mpc^3

$$\xi(r) = \frac{1}{(2\pi)^3} \int P(\vec{k})\ e^{+i\vec{k}\cdot\vec{r}} d^3\vec{k}$$

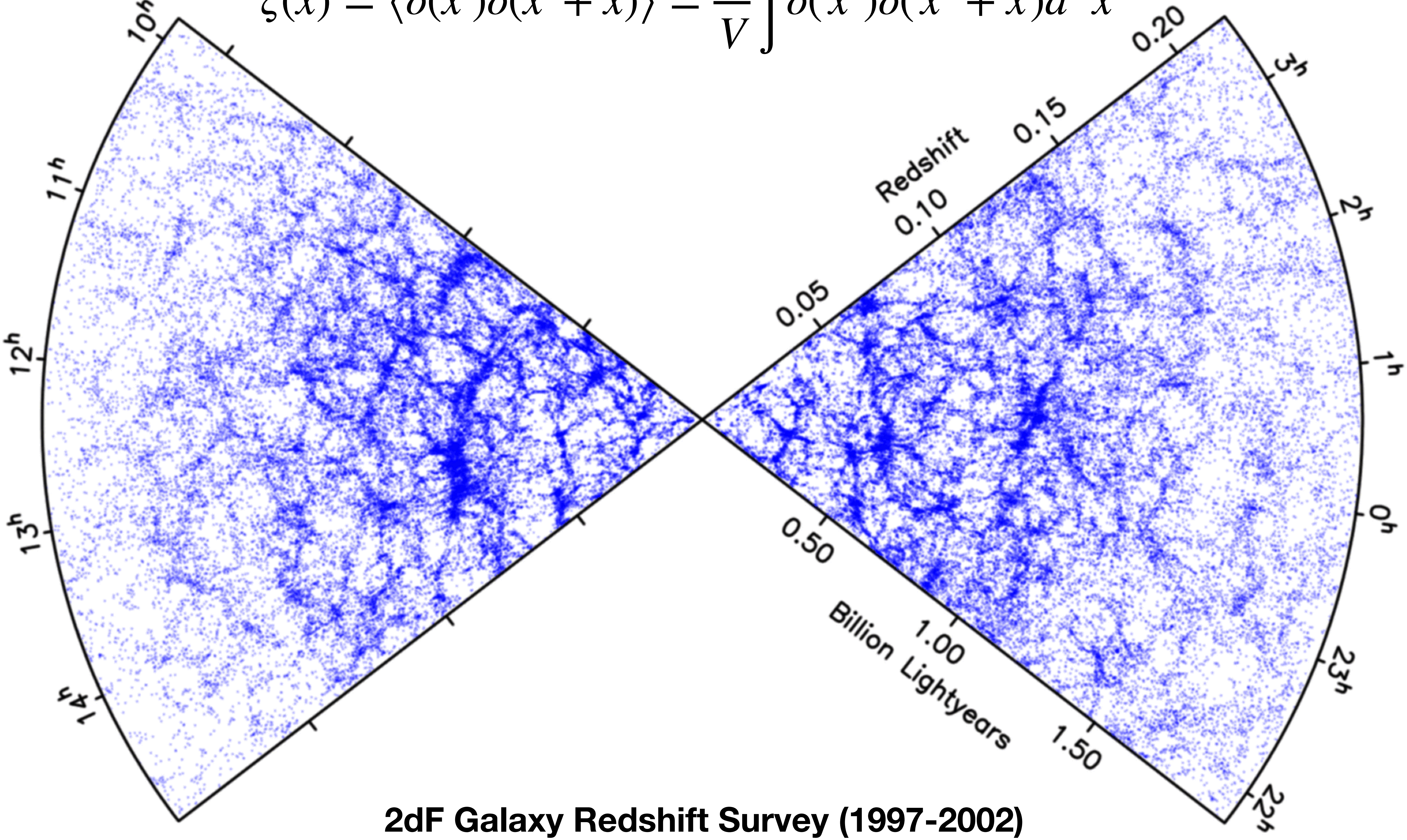
dimensionless

The F.T. of Power Spectrum is the two-point correlation function

$$\xi(r) = \langle \delta(\vec{x})\delta(\vec{x}+\vec{r}) \rangle = \frac{1}{V} \int \delta(\vec{x})\delta(\vec{x}+\vec{r}) d^3\vec{x}$$

Estimate correlation function from galaxy surveys

$$\xi(x) = \langle \delta(x')\delta(x'+x) \rangle = \frac{1}{V}\int \delta(\overrightarrow{x'})\delta(\overrightarrow{x'}+\vec{x})d^3\overrightarrow{x'}$$



2dF Galaxy Redshift Survey (1997-2002)

Estimate correlation function from galaxy surveys

$\xi(r)$ is a measure of the excess probability of finding another galaxy at distance r above expectation from an unclustered random distribution:

$$dP = n[1 + \xi(r)]dV,$$

It can be measured by counting pairs of galaxies as a function of separation and divides by expectation from random unclustered mock catalogs:

$$\xi = \frac{n_R}{n_D}\frac{DD}{DR} - 1$$

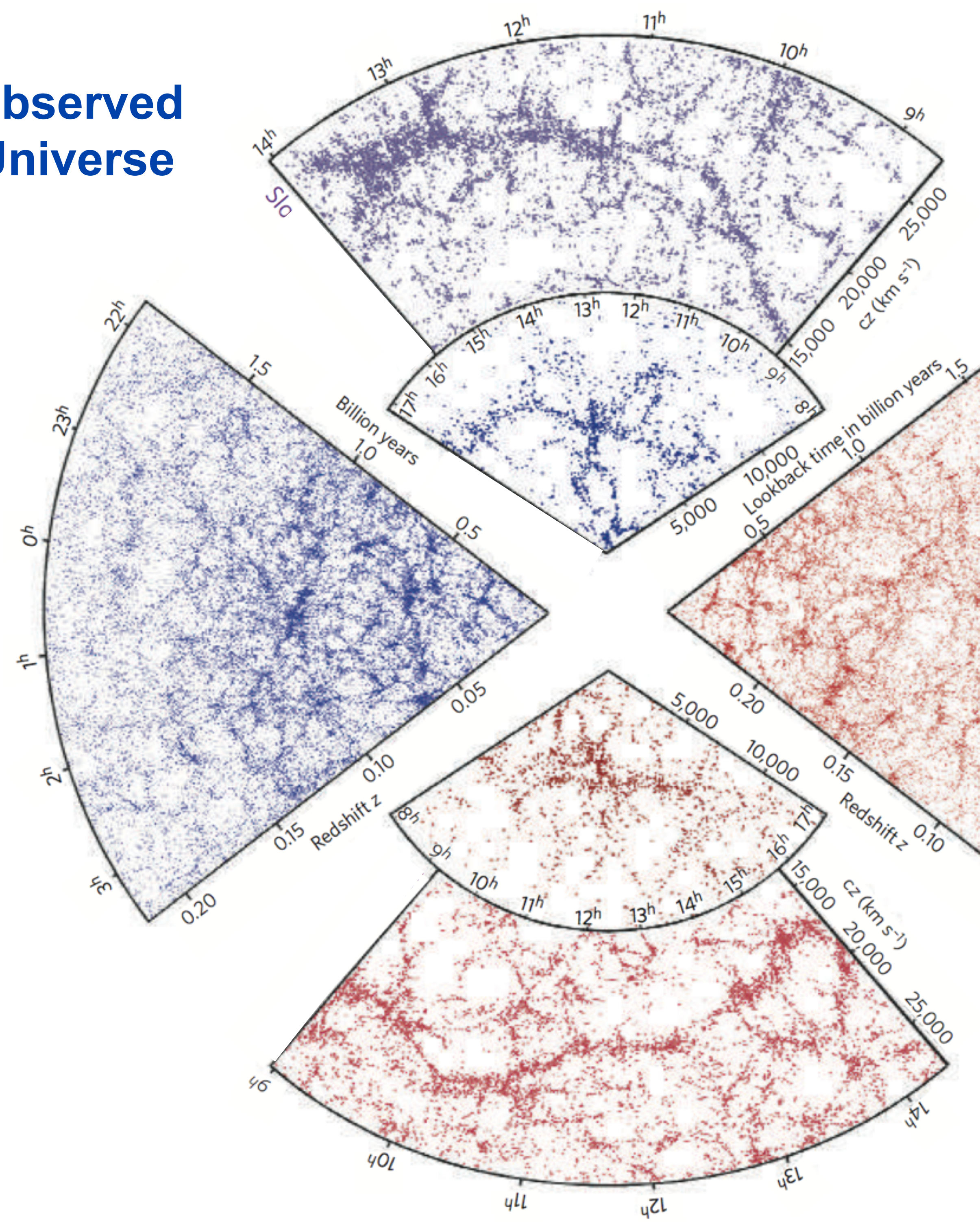
$$\xi = \frac{1}{RR}\left[DD\left(\frac{n_R}{n_D}\right)^2 - 2DR\left(\frac{n_R}{n_D}\right) + RR\right]$$

In the above two popular estimators (Davis & Peebles 1983 and Landy & Szalay 1993), *DD, DR, and RR* are counts of galaxy pairs within a range of separation in the data catalog and between the data and mock catalog, and between two different mock catalogs, and n_D and n_R are the mean number densities of galaxies in the data and the mock catalogs.

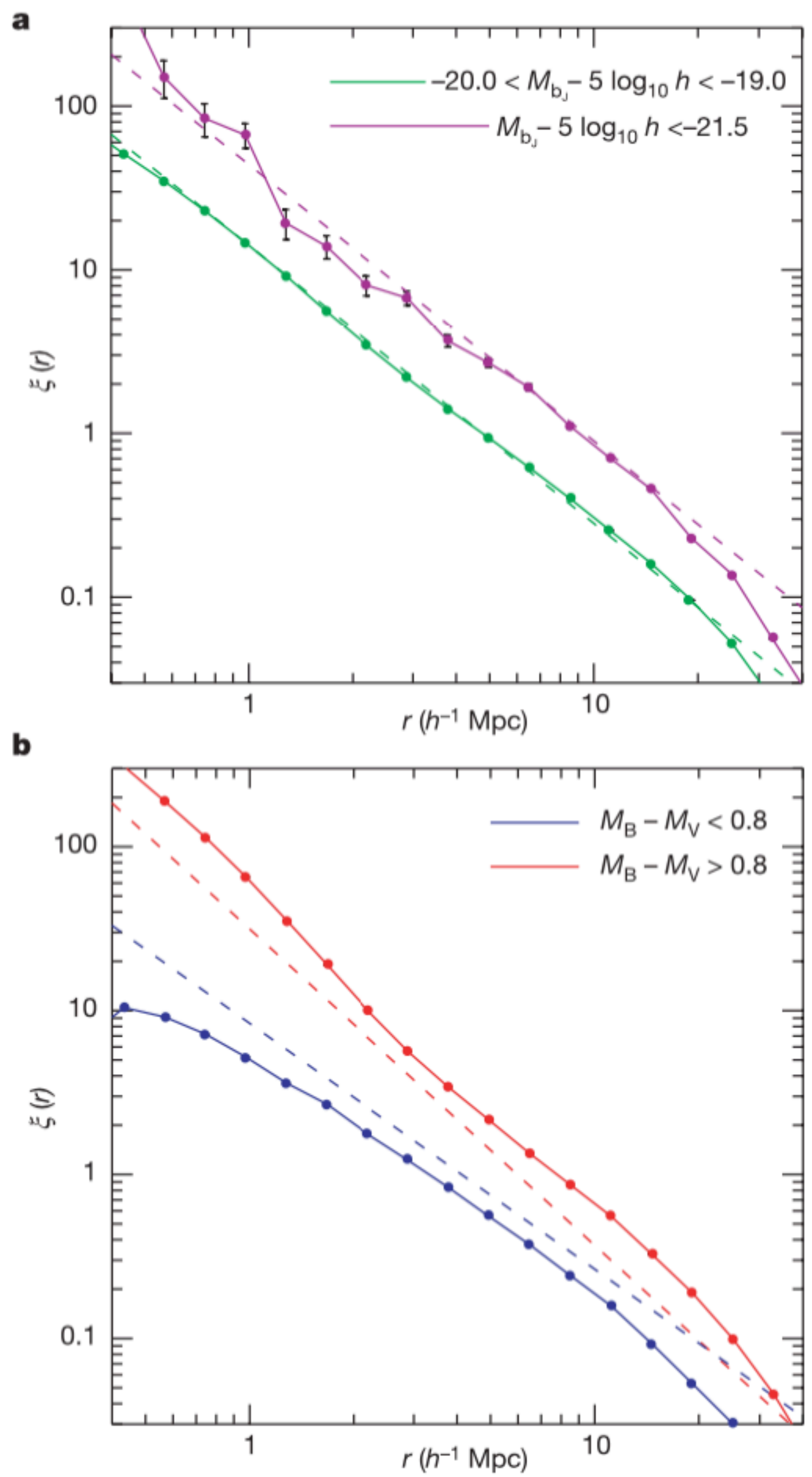
Correlation Functions of Galaxies w/ Different Masses
“more massive galaxies are more clustered”

Observed Universe

A



B



Simulated Universe (DM only)

Springel et al. (2005, 2006)

Evolution of the Power Spectrum

$$\sigma_M^2 = \frac{1}{2\pi^2}\int_0^\infty P(k)W_M^2(k)k^2dk$$

How power spectrum change spatial correlations?

Define **two-point correlation function**:

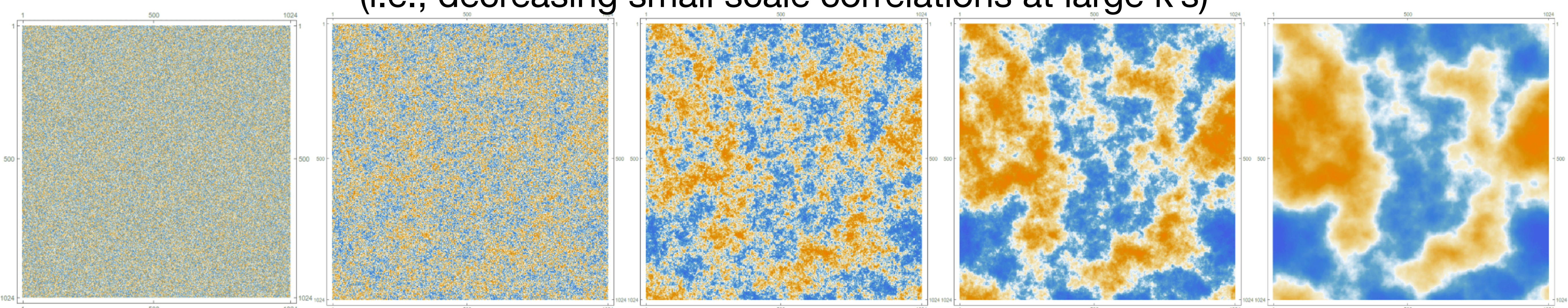
$$\xi(\vec{x}) = \langle \delta(\vec{x}')\delta(\vec{x}' + \vec{x}) \rangle = \frac{1}{V}\int \delta(\vec{x}')\delta(\vec{x}' + \vec{x})d^3\vec{x}'$$

its **Fourier transform** is called the **power spectrum:**

$$P(\vec{k}) = V\Delta^2(\vec{k})$$

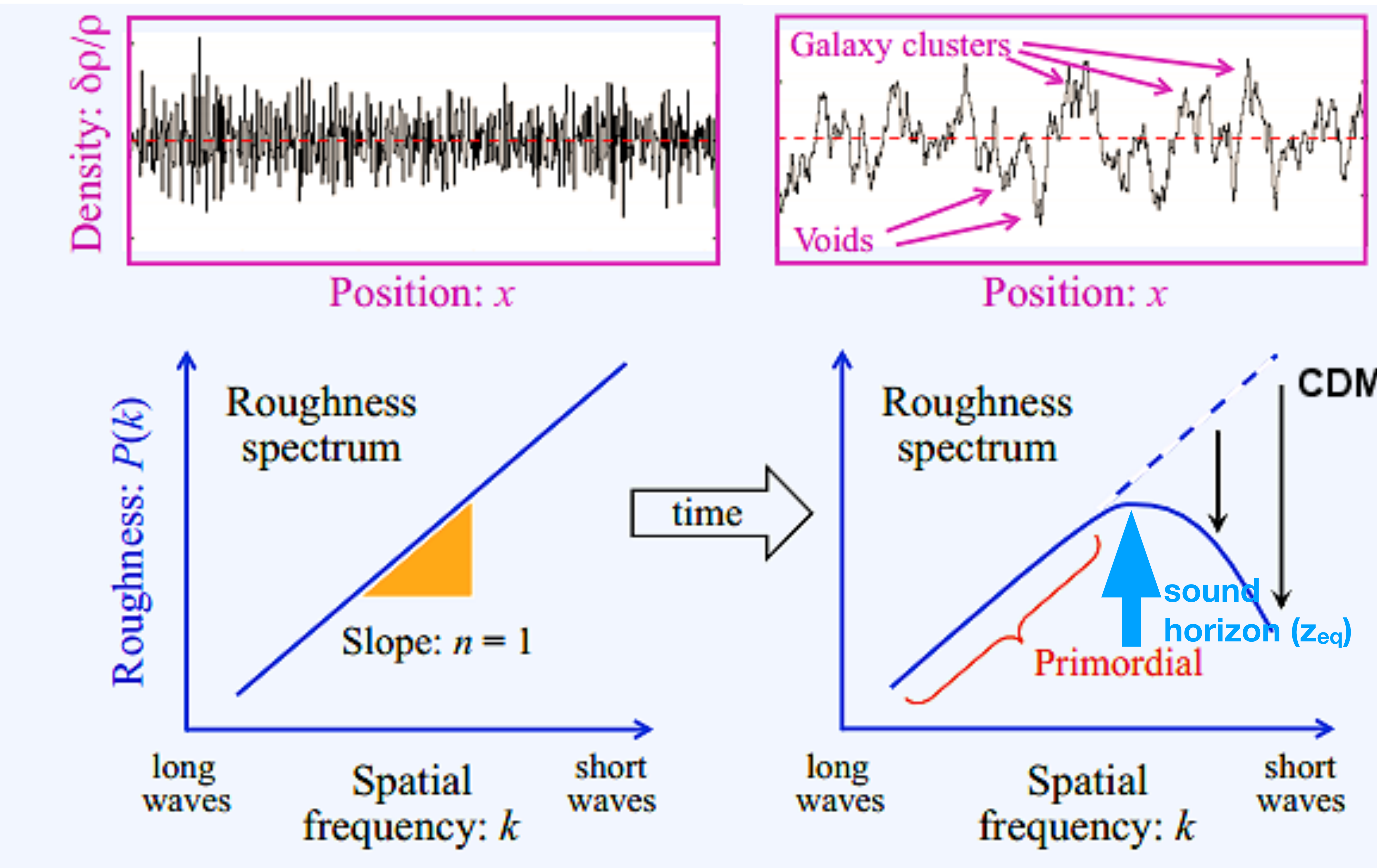
$$\text{where } \Delta(\vec{k}) = \int \delta(\vec{x})\ e^{-i\vec{k}\cdot\vec{x}}d^3\vec{x}$$

Gaussian random fields w/ ever steeper power spectrum
(i.e., decreasing small scale correlations at large k's)



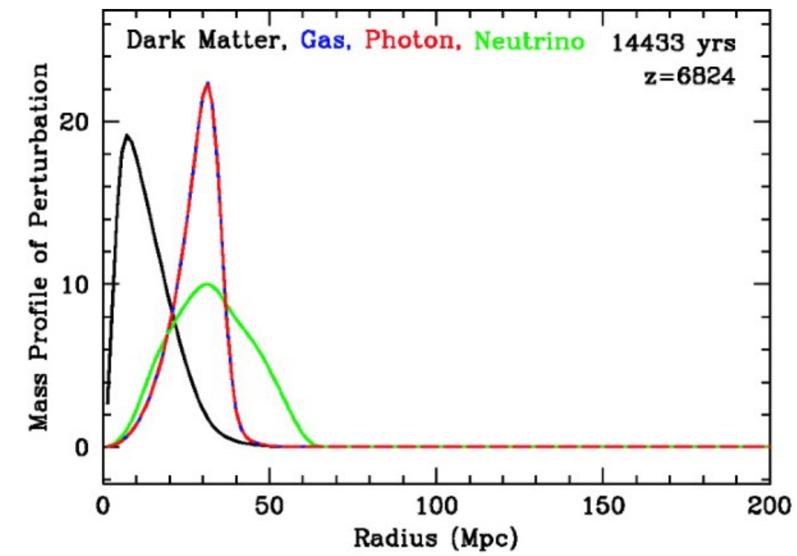
$P(k) = k^0$ $\quad$ $P(k) = k^{-1}$ $\quad$ $P(k) = k^{-2}$ $\quad$ $P(k) = k^{-3}$ $\quad$ $P(k) = k^{-4}$

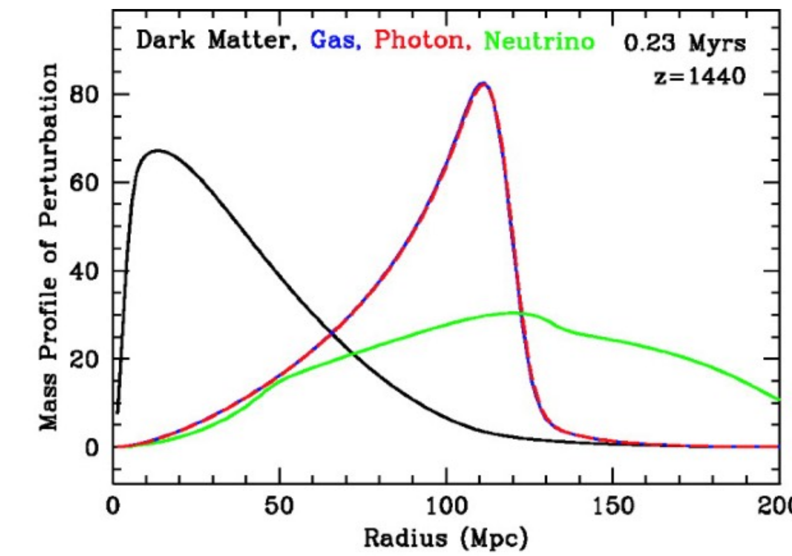
Evolution of correlation function <-> evolution of power spectrum from primordial density perturbations to the epoch of recombination



Evolution of the radial mass profile (comoving) of an initially pointlike overdensity

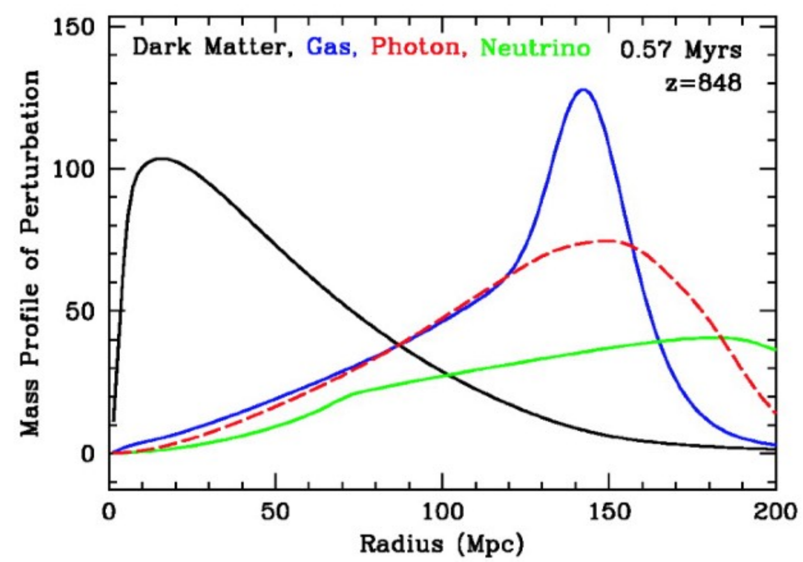
Near the initial time, the photons and baryons travel outward as a pulse.

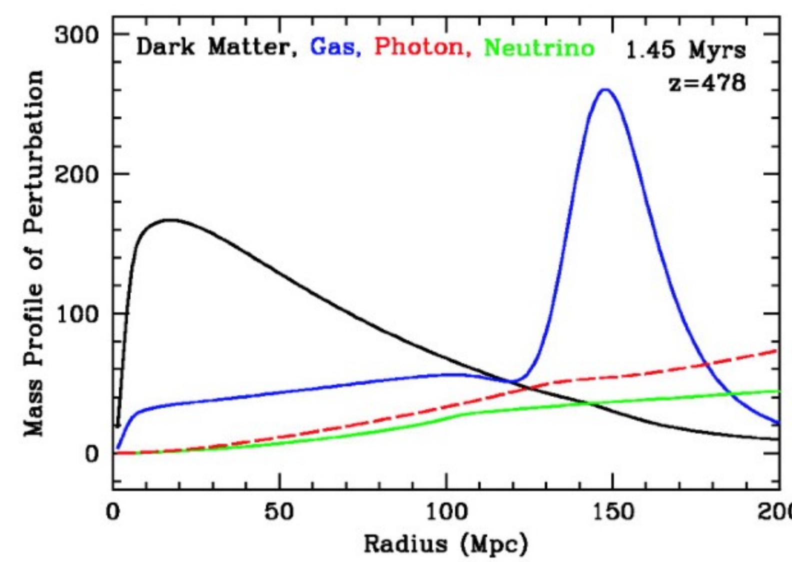




A wake in the cold dark matter is raised by the outward-going pulse of baryons and relativistic species

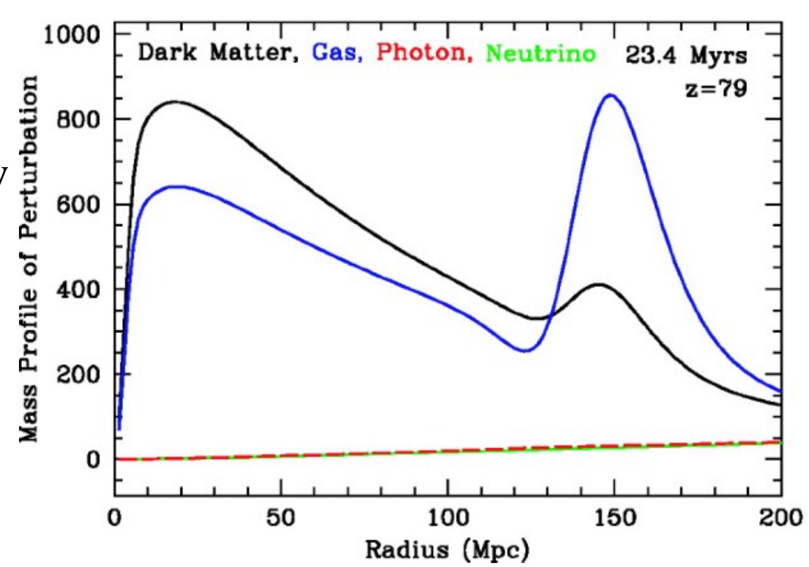
Around **recombination**, the photons leak away from the baryonic perturbation

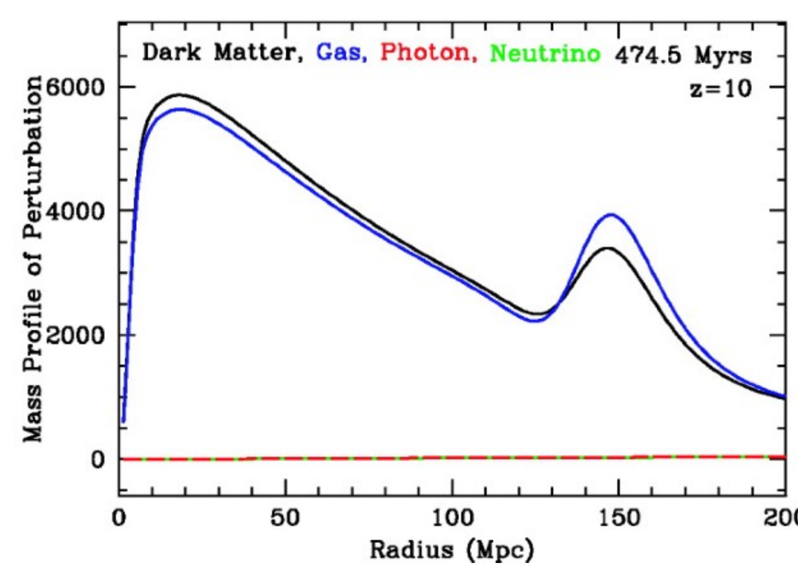




After **recombination**, a CDM perturbation toward the center and a baryonic perturbation in a shell

Gravitational instability now takes over, and new baryons and dark matter are attracted to the two overdensities (center & outer shell)





At late times ($z<10$), the baryonic fraction of the perturbation is near the cosmic value

Eisenstein+2007 calculated using CMBFAST (predecessor of CAMB)

Eisenstein et al. 2005: detection of the Baryon Acoustic Peak from SDSS luminous red galaxies

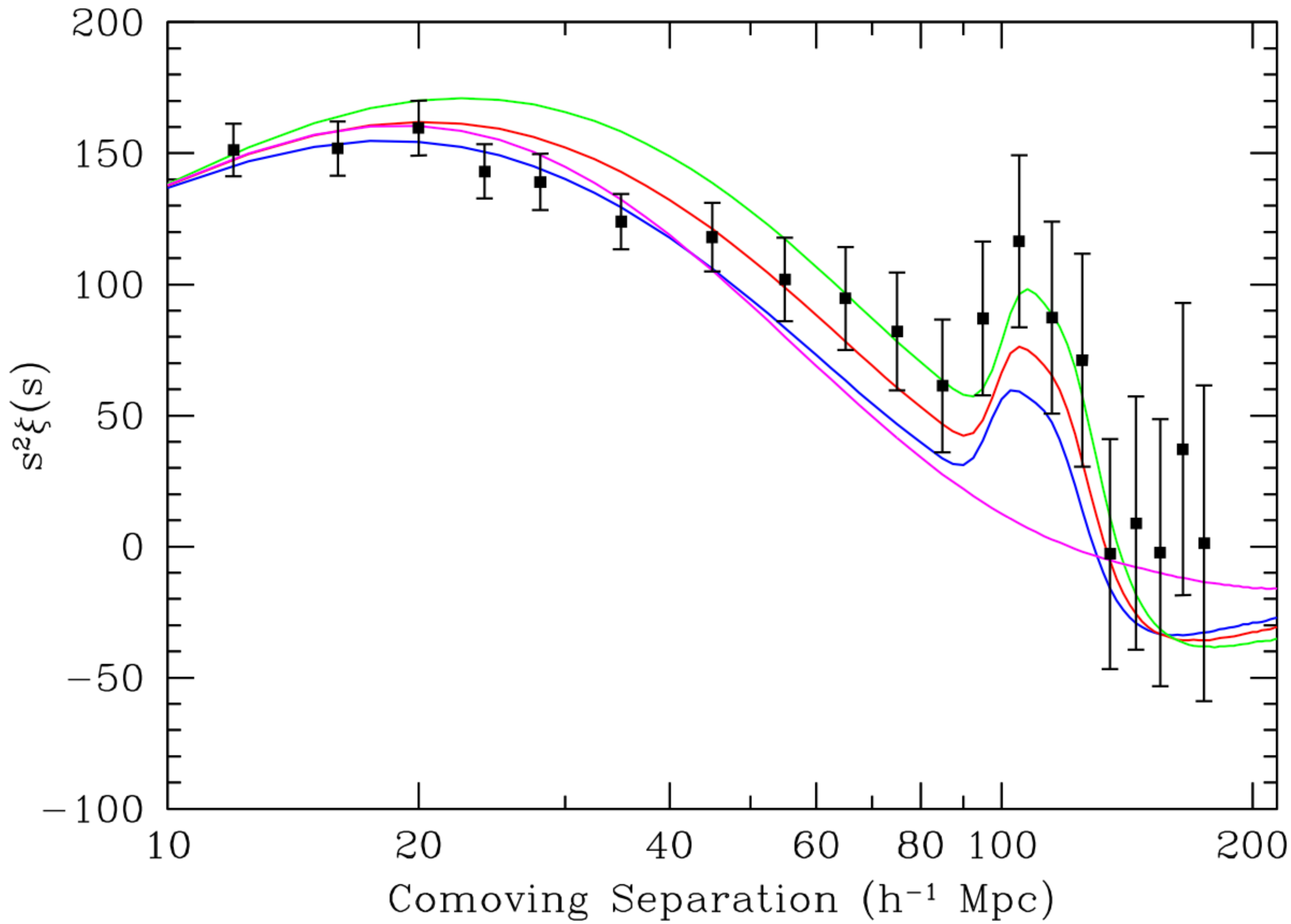
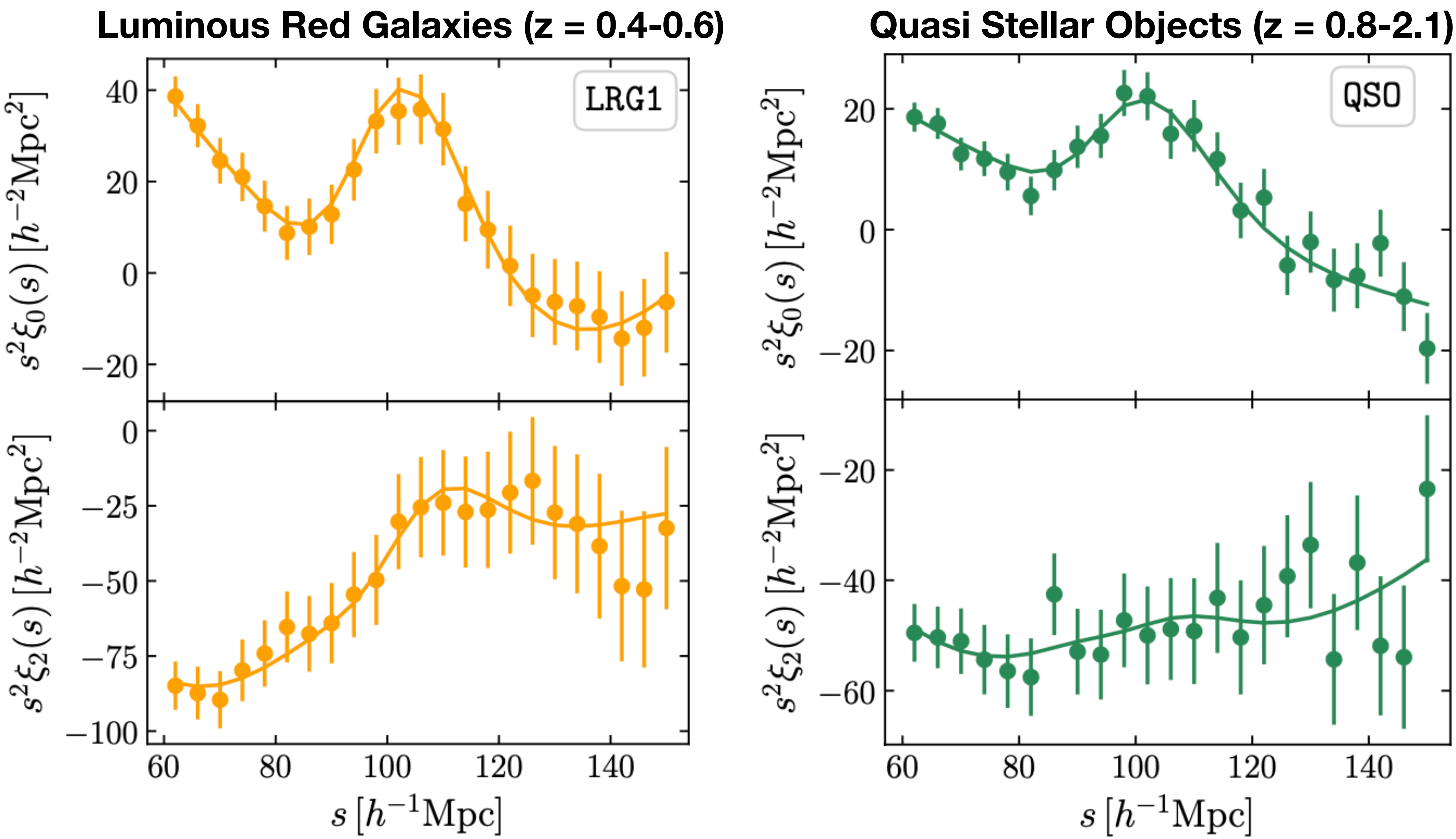
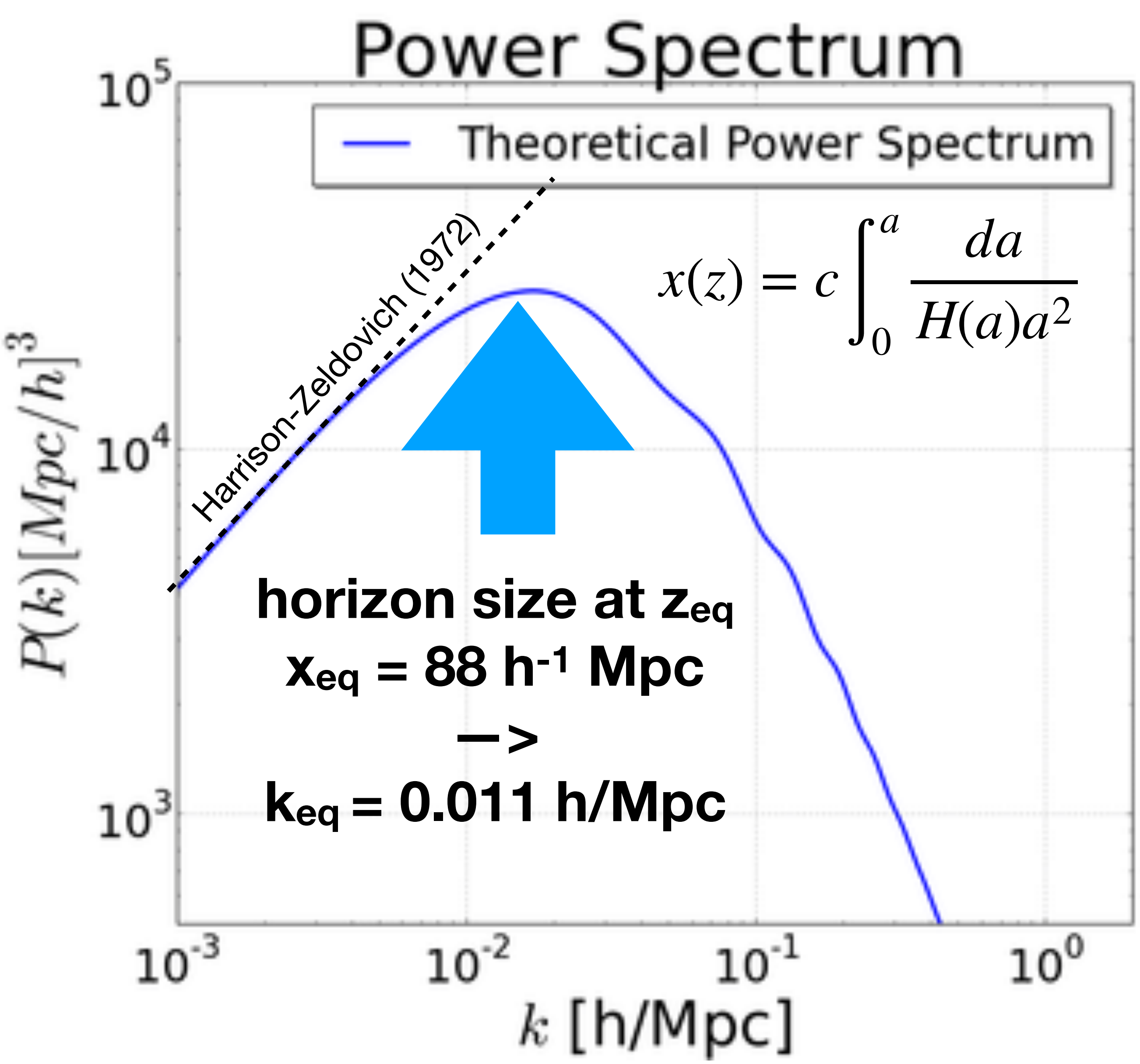


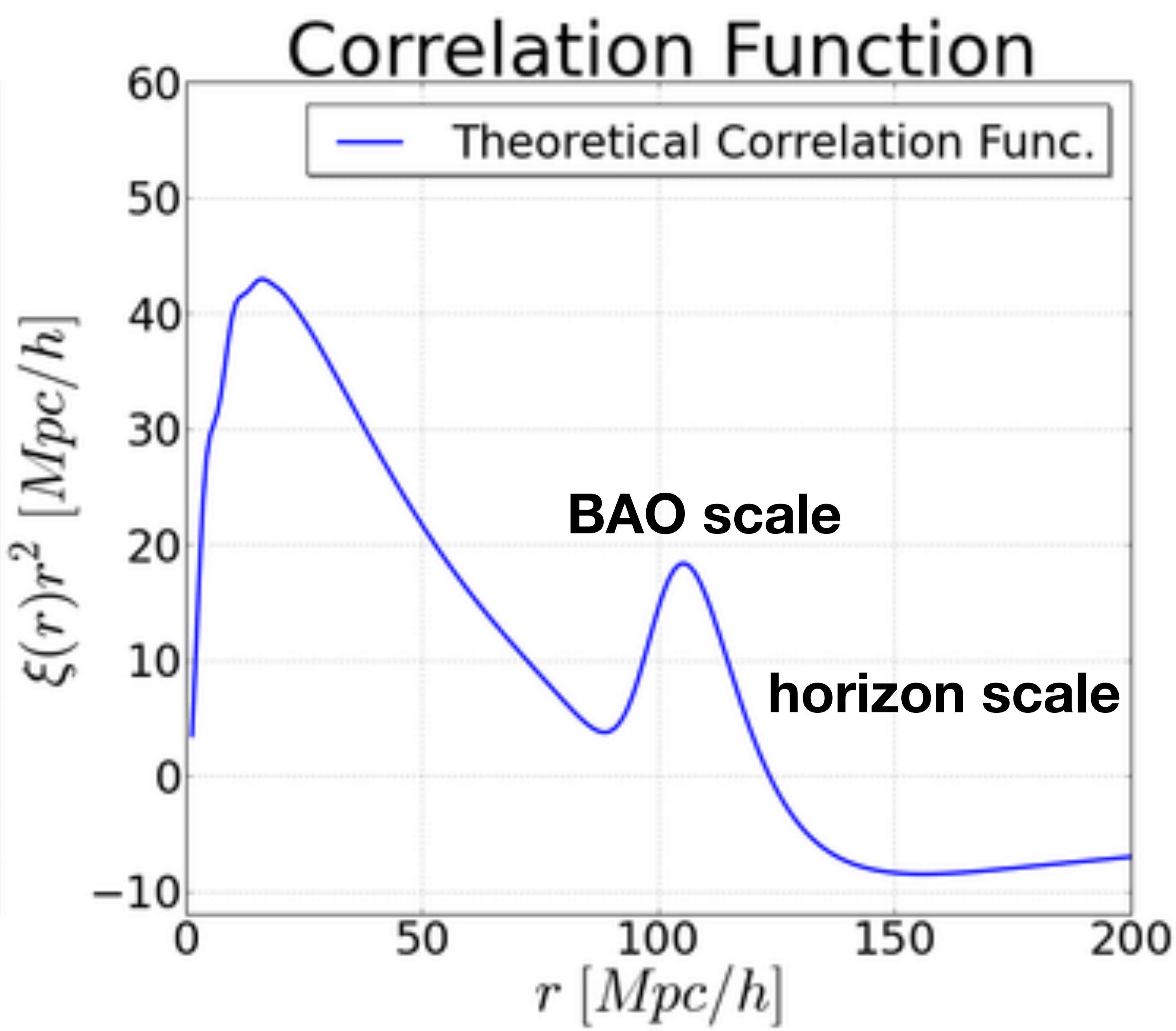
FIG. 3.— As Figure 2, but plotting the correlation function times s^2. This shows the variation of the peak at $20h^{-1}$ Mpc scales that is controlled by the redshift of equality (and hence by $\Omega_m h^2$). Varying $\Omega_m h^2$ alters the amount of large-to-small scale correlation, but boosting the large-scale correlations too much causes an inconsistency at $30h^{-1}$ Mpc. The pure CDM model (magenta) is actually close to the best-fit due to the data points on intermediate scales.

DESI DR2 Correlation Functions (DESI 2025 March)

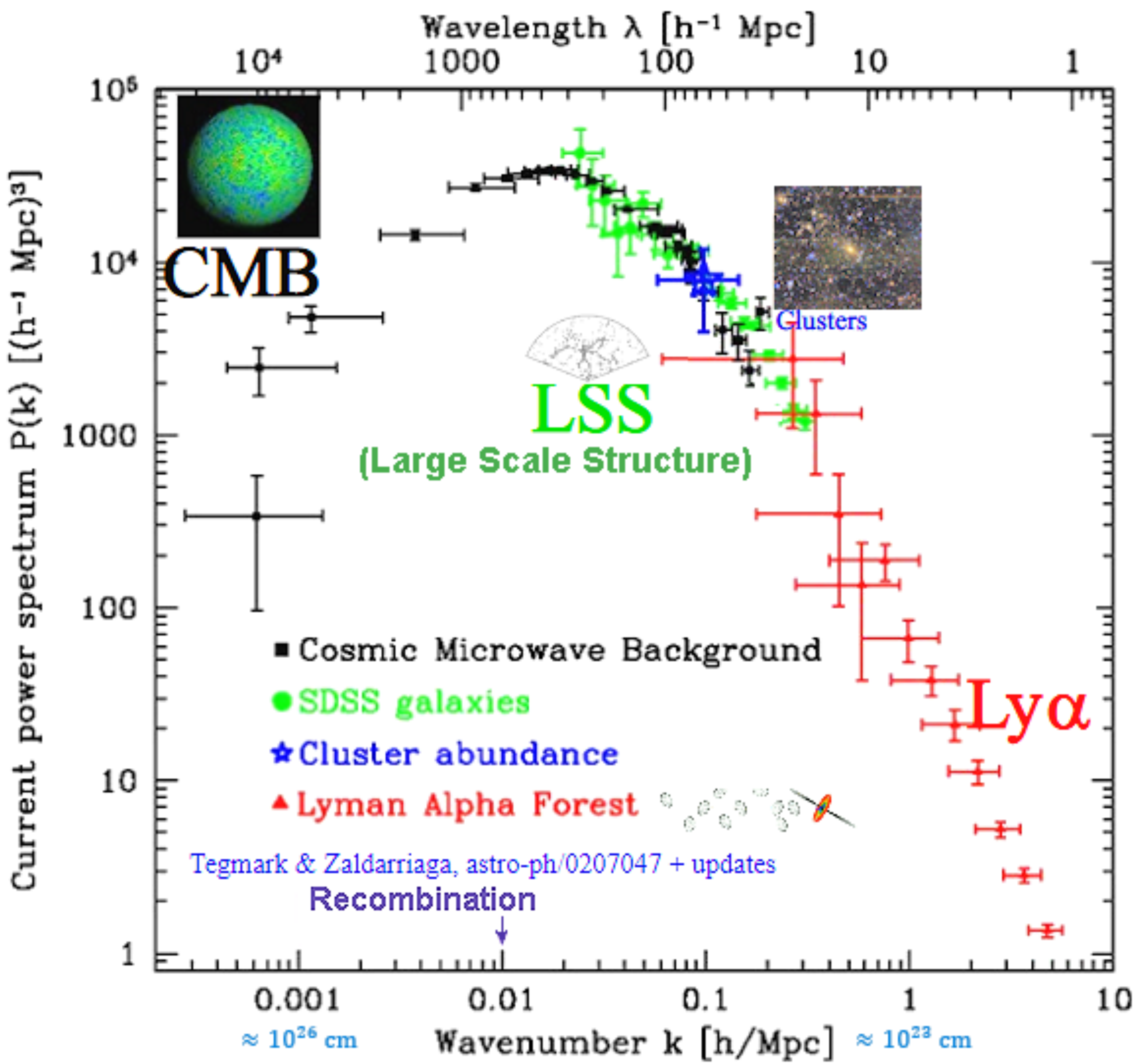


Evolution of the linear power spectrum and correlation function in the LCDM universe





https://lambda.gsfc.nasa.gov/toolbox/tb_camb_form.cfm

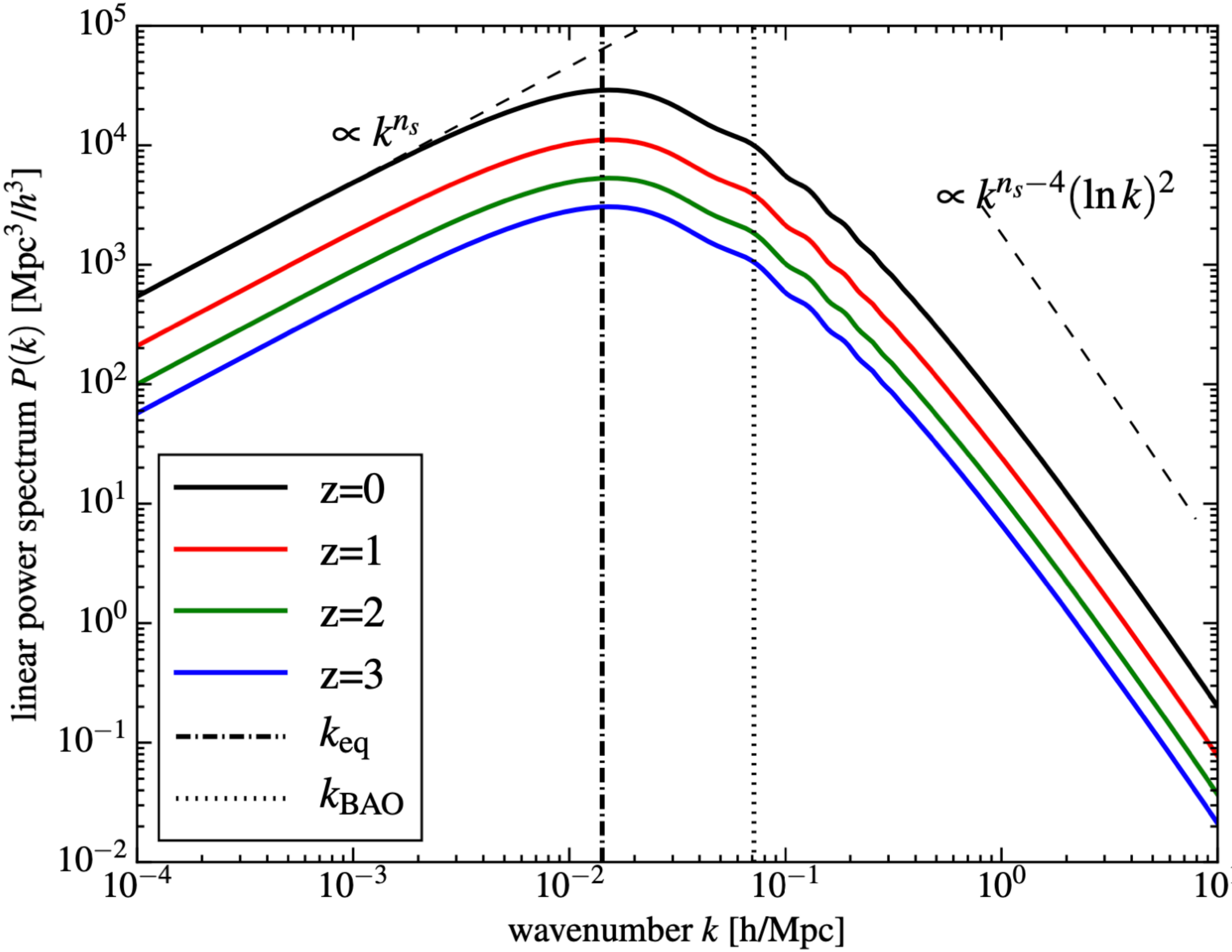


power spectrum

measured from correlation functions of galaxies and the CMB

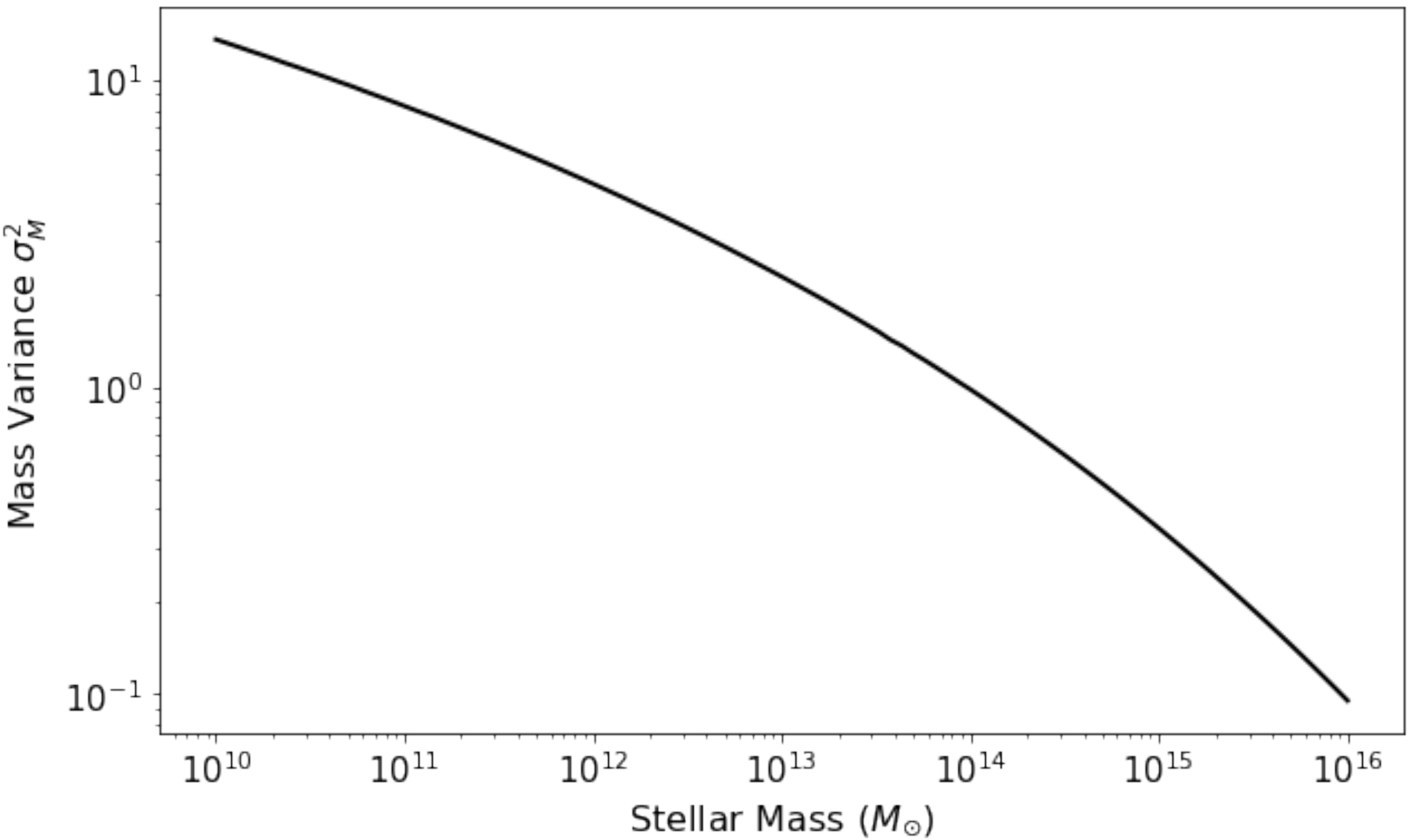
Calculation of Halo Mass Function

mass variance from integrating the power spectrum: a. power spectra at different redshifts



mass variance from integrating the power spectrum: b. mass variance vs. M at z = 0

$$\sigma_M^2 = \frac{1}{2\pi^2}\int_0^\infty P(k)W_M^2(k)k^2dk$$



Homework Problem: approximating FT of a top hat with another top hat

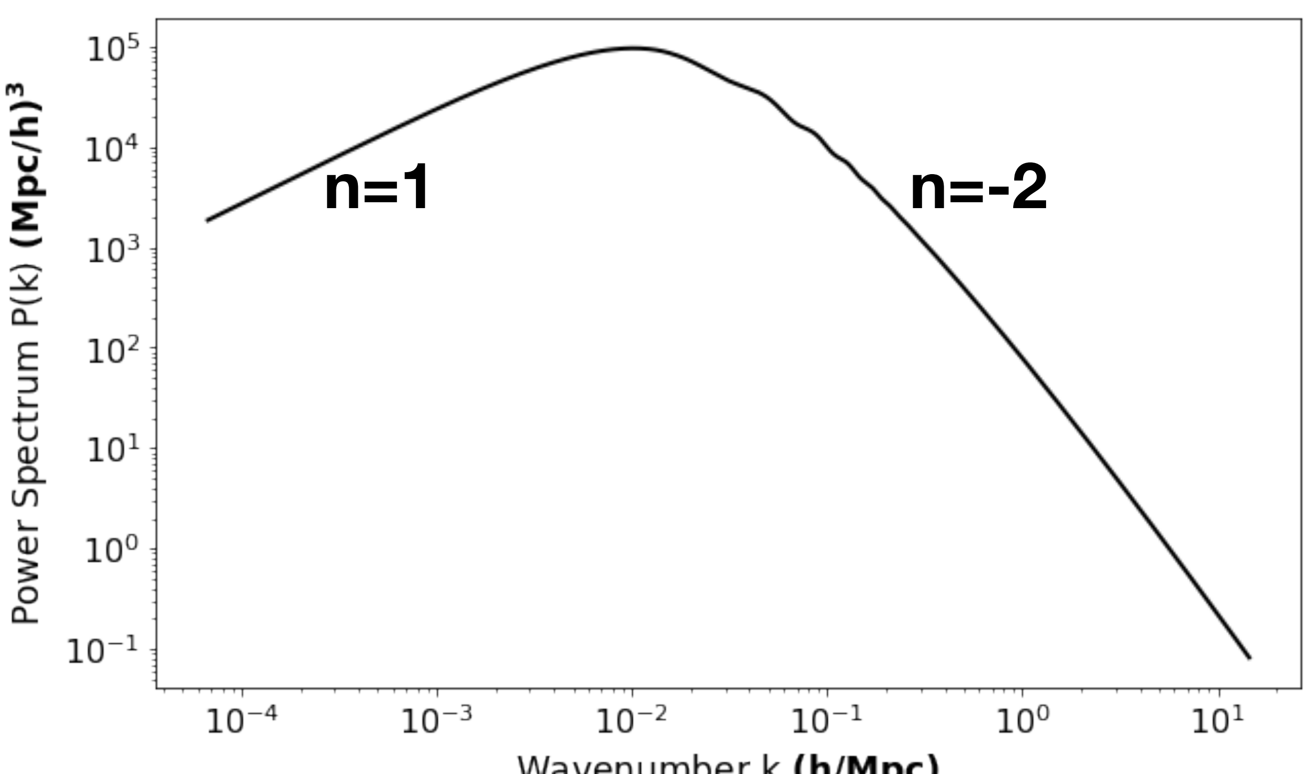
Approximating the window function as a **top-hat** in **frequency**:

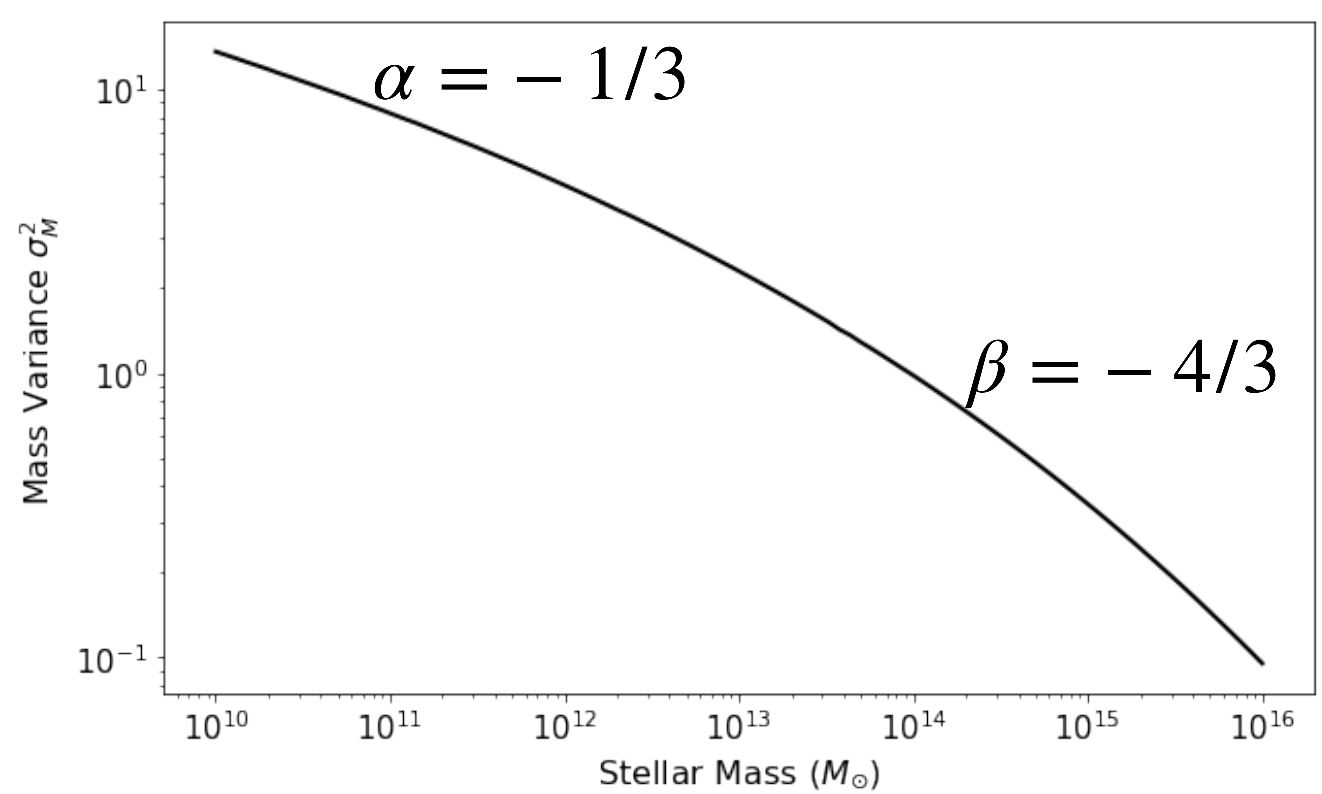
$$\sigma_M^2 = \frac{1}{2\pi^2}\int_0^\infty P(k)W_M^2(k)k^2dk \approx \frac{1}{2\pi^2}\int_0^{2\pi/R} P(k)k^2dk$$

When the **power spectrum** is a power law: $P(k) \propto k^n$:

$$\sigma_M^2 \approx \frac{1}{2\pi^2}\int_0^{2\pi/R} P(k)k^2dk \propto R^{-(n+3)}$$

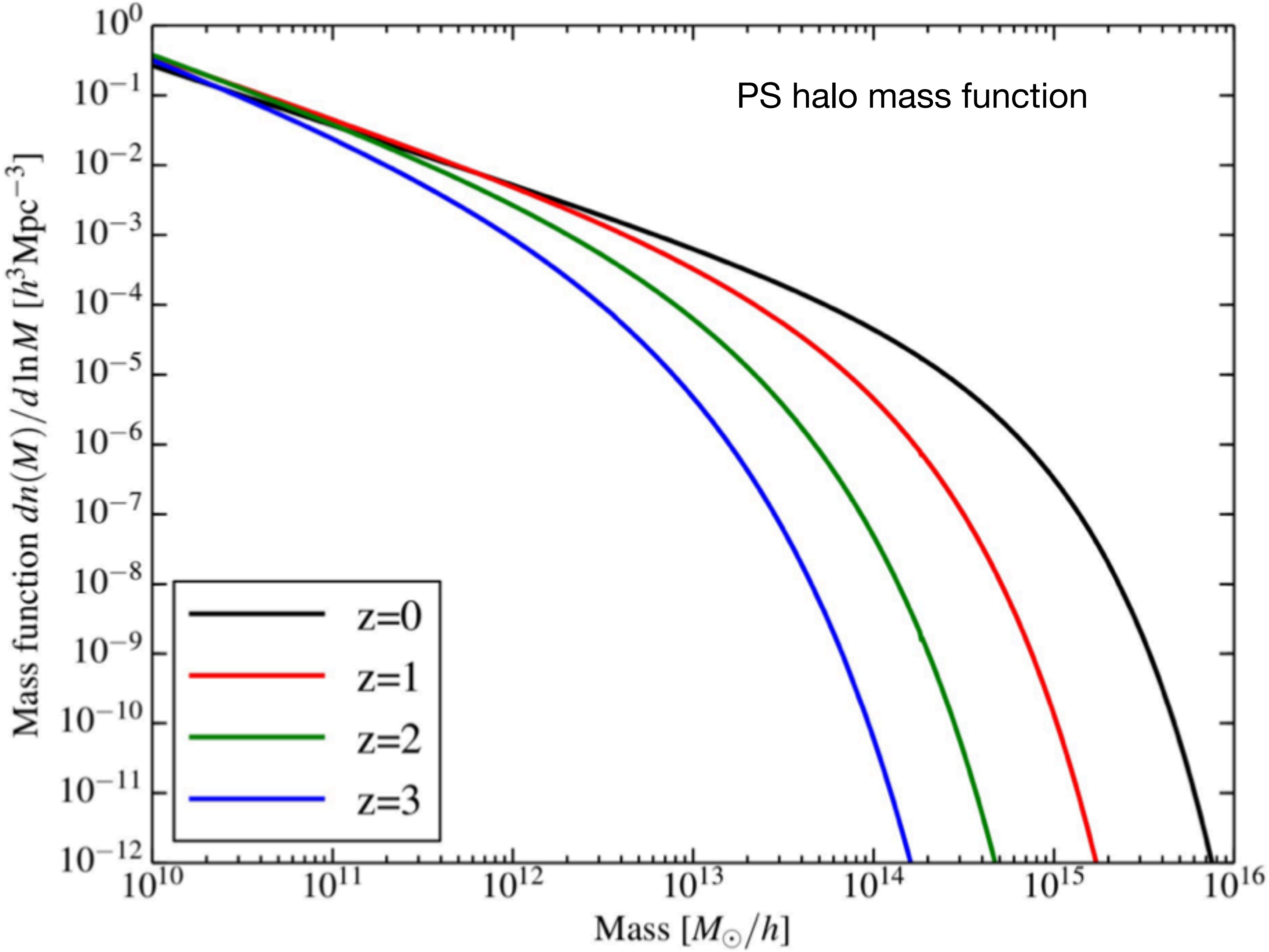
$$\sigma_M^2 \propto M^{-(n+3)/3} \text{ when } P(k) \propto k^n$$





Resulting PS Halo Mass Function at Various Redshifts

$$\phi(M,z) = \sqrt{\frac{2}{\pi}}\frac{\bar{\rho}}{M^2}\frac{\delta_c}{\sigma_M}\left|\frac{d\ln\sigma_M}{d\ln M}\right|\exp\left(-\frac{\delta_c^2}{2\sigma_M^2}\right)$$



Summary: PS Halo Mass Function

The PS differential halo mass function:

$$\phi(M, z) = \sqrt{\frac{2}{\pi}} \frac{\bar{\rho}}{M^2} \frac{\delta_c}{\sigma_M} \left| \frac{d \ln \sigma_M}{d \ln M} \right| \exp\left(-\frac{\delta_c^2}{2\sigma_M^2} \right)$$

The collapse threshold:

$$\delta_c = 1.686(1 + z)$$

The mass variance from power spectrum:

$$\sigma_M^2 = \frac{1}{2\pi^2} \int_0^\infty P(k) W_M^2(k) k^2 dk \approx \frac{1}{2\pi^2} \int_0^{2\pi/R} P(k) k^2 dk$$

if the power spectrum is a power law: $P(k) \propto k^n$:

$$\sigma_M^2 \approx \frac{1}{2\pi^2} \int_0^{2\pi/R} P(k) k^2 dk \propto R^{-(n+3)} \propto M^{-(n+3)/3}$$

The power spectrum and the two point correlation function:

$$P(\vec{k}) \equiv V\Delta^2(\vec{k}) = \int \xi(\vec{r})\, e^{-i\vec{k}\cdot\vec{r}} d^3\vec{r}$$

$$\xi(\vec{r}) = \langle \delta(\vec{x})\delta(\vec{x} + \vec{r}) \rangle = \frac{1}{V} \int \delta(\vec{x})\delta(\vec{x} + \vec{r}) d^3\vec{x}$$

$$\phi(M,z) = \sqrt{\frac{2}{\pi}} \frac{\bar{\rho}}{M^2} \frac{\delta_c}{\sigma_M} \left| \frac{d \ln \sigma_M}{d \ln M} \right| \exp\left(-\frac{\delta_c^2}{2\sigma_M^2} \right) \quad \text{-->} \quad \phi(M) = \frac{\mathrm{d}N}{\mathrm{d}V\mathrm{d}M} = \frac{\phi_\star}{M_\star} \left(\frac{M}{M_\star} \right)^{-1.5} \exp\left(-\frac{M}{M_\star} \right)$$

If we define a characteristic mass at redshift z, $M_\star(z)$, by requiring

$$2\sigma_M^2[M_\star(z)] = \delta_c^2(z) = [1.686(1+z)]^2,$$

we'll see the behavior of the predicted mass function at the two extremes:

- When $M \ll M_\star$, $\sigma_M \gg \delta_c$, the exponential component approaches unity. If $\sigma_M \propto M^{-\alpha}$, then $\sigma_M = \delta_c (M/M_\star)^{-\alpha}/\sqrt{2}$, and $\phi(M) \propto (M/M_\star)^{\alpha-2}$

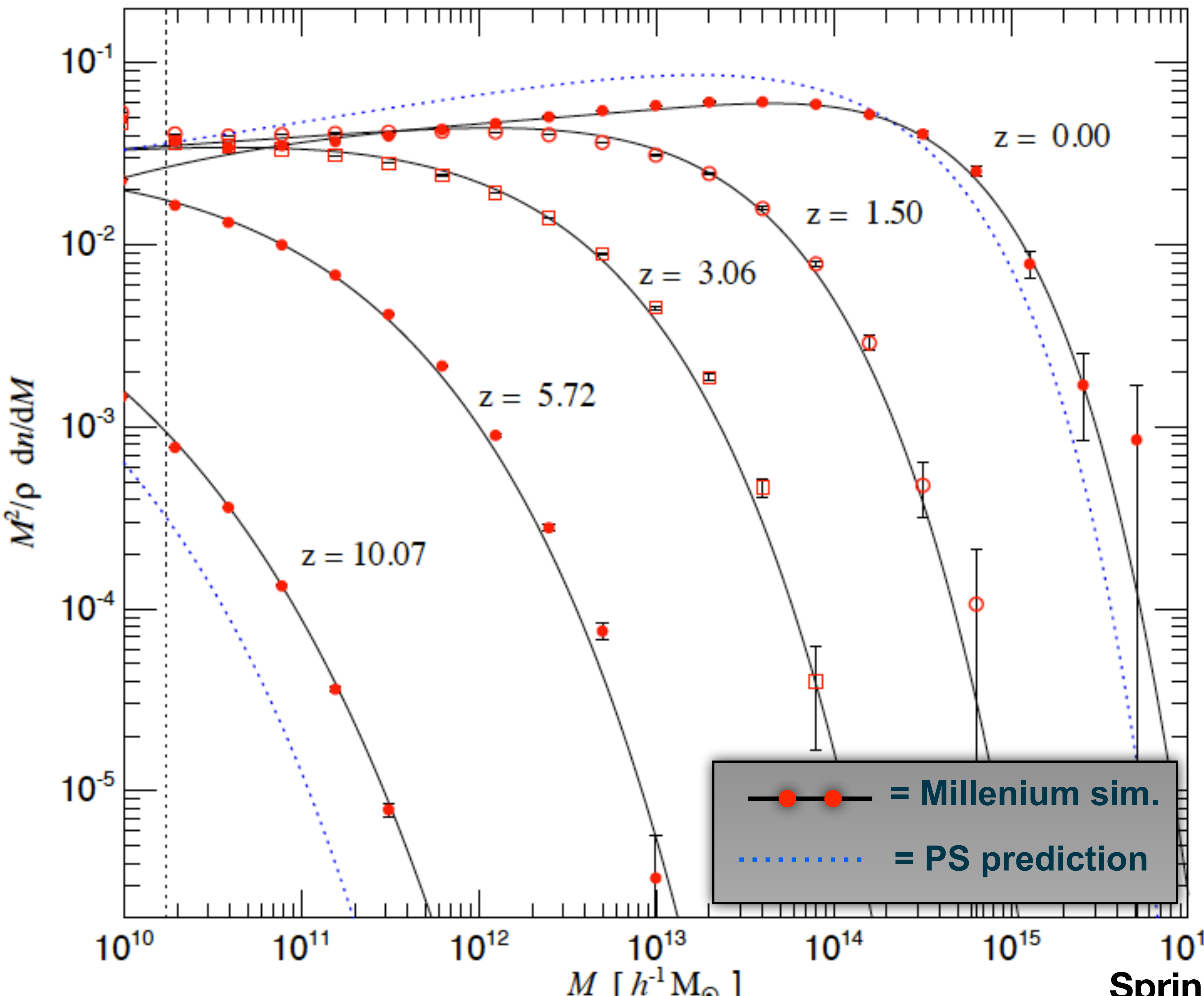
- When $M \gg M_\star$, $\sigma_M \ll \delta_c$, the exponential component becomes important. If $\sigma_M \propto M^{-\beta}$, then $2\sigma_M^2 = \delta_c^2 (M/M_\star)^{-\beta}$, and $\phi(M) \propto \exp[-(M/M_\star)^\beta]$

- if the power spectrum is a power law, $P(k) \propto k^n$, then $\sigma_M \propto M^{-(n+3)/6}$

  - When $M \ll M_\star$, k is large and $P(k) \propto k^{-2}$ so that $\sigma_M \propto M^{-1/6}$, $\phi(M) \propto (M/M_\star)^{-11/6}$

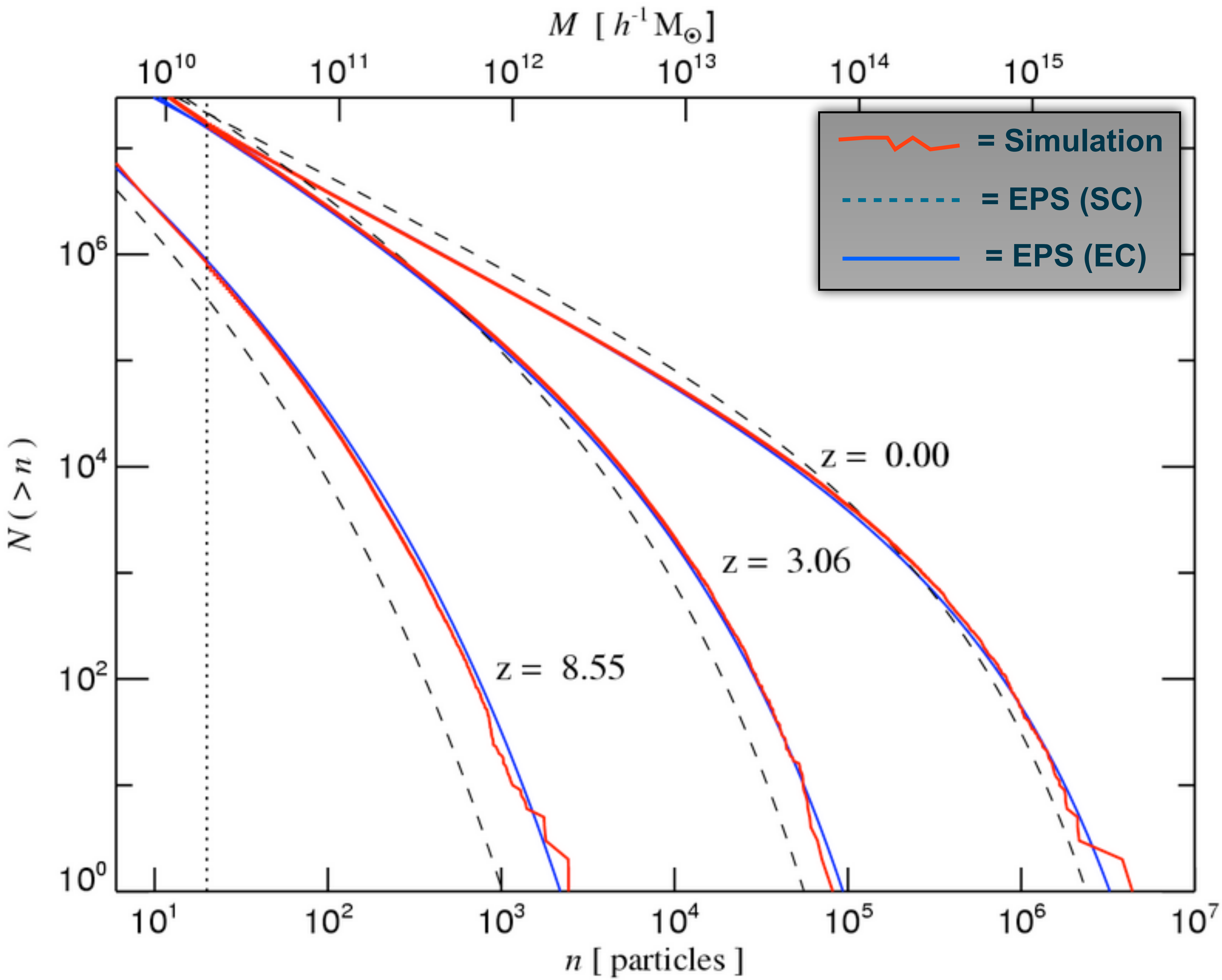
  - When $M \gg M_\star$, k is small and $P(k) \propto k$, so that $\sigma_M \propto M^{-2/3}$, $\phi(M) \propto \exp[-(M/M_\star)^{2/3}]$

Halo Mass Functions: N-body Simulation vs. Analytical Predictions



Springel+2005

Extended Press-Schechter formalism improves the agreement w/ N-body simulations



Source: Millenium Simulation; Springel V., 2004 (MPA research highlight)

How well do PS or N-body predicted **halo** mass functions match observed **galaxy** mass functions?

Problem: both the P-S and N-body halo mass function strongly disagrees w/ *Observed* galaxy mass function