Density Profiles of DM Haloes

① Isothermal sphere (Singular)

$$\rho(r) = \frac{\sigma^2}{2\pi G r^2}, \quad M(r) = \frac{2\sigma^2 r}{G}, \quad \phi(r) = 2\sigma^2 \ln(r) + \text{const.}$$

This is an isotropic model w/ a dist. function

$$f(E) = \frac{\rho_0}{(2\pi\sigma^2)^{3/2}} e^{-E/\sigma^2}, \quad E = \frac{1}{2}v^2 + \phi(r)$$

$$\Rightarrow \rho(\Phi) = \int_0^{v_{esc}} 4\pi v^2 f(E)\, dv = 4\pi \int_{\Phi}^{0} f(E)\sqrt{2E - 2\Phi}\, dE = \rho_0 \exp\left(-\frac{\Phi}{\sigma^2}\right)$$

$$\Rightarrow \Phi = -\sigma^2 \ln(\rho/\rho_0) \quad \Rightarrow d\Phi = -\sigma^2 d\ln\rho$$

plug this to Poisson Eq. and solve for ρ $(\nabla^2\Phi = 4\pi G\rho)$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\Phi}{dr}\right) = 4\pi G\rho,$$

$$\Rightarrow \frac{d}{dr}\left(r^2 \frac{d\ln\rho}{dr}\right) = -\frac{4\pi G}{\sigma^2}\rho \cdot r^2 \quad \Rightarrow \rho(r) = \frac{\sigma^2}{2\pi G r^2}$$

why this density profile is called isothermal?

consider isothermal gas in hydrostatic equilibrium:

$$\frac{dP}{dr} = \frac{kT}{m}\cdot\frac{d\rho}{dr} = -\rho\frac{d\Phi}{dr} = -\rho\frac{GM(r)}{r^2}$$

multiply $\frac{mr^2}{\rho kT}$ on both sides & diff. by r

$$\frac{d}{dr}\left(r^2 \frac{d\ln\rho}{dr}\right) = -\frac{4\pi G m}{kT} r^2\rho, \quad \text{where we used } \frac{dM}{dr} = 4\pi\rho r^2$$

define $\sigma^2 = kT/m$ we have the same equation as the Poisson Eq.

thus, self-gravitating isothermal gas will have a density profile ($\rho \propto r^{-2}$) same as self-gravitating collisionless particles that have a D.F. $f(E)$ above.

An important property of SIS is its flat rotation curve:

$$V_{circ} = \sqrt{\frac{GM(r)}{r}} = \sqrt{2}\,\sigma = \text{const.}$$

② Navarro-Frenk-White (NFW 1996)

based on analysis of density profiles formed in DM-only simulations

$$\rho(r) = \frac{4\rho_s}{(r/r_s)(1+r/r_s)^2} \quad , \; r_s : \text{scale radius} , \; \rho_s = \rho(r_s)$$

$$\Rightarrow M(r) = \int_0^r 4\pi r^2 \rho(r)\, dr = 16\pi \rho_s r_s^3 \left[\ln\left(1+\frac{r}{r_s}\right) - \frac{r/r_s}{1+r/r_s} \right]$$

$$\Rightarrow \rho_s = \frac{M_\Delta}{16\pi r_s^3 [\ln(1+c_\Delta) - c_\Delta/(1+c_\Delta)]} \quad , \; c_\Delta = \frac{r_\Delta}{r_s} \; \text{concentration}$$

$$\Phi(r) = -16\pi G \rho_s r_s^2 \frac{\ln(1+r/r_s)}{r/r_s}$$

$$\boxed{\begin{array}{l} \text{if we define SIS density profile as :} \\ \rho(r) = \rho_s \cdot (r/r_s)^{-2} \quad \text{then} \quad \sigma^2 = 2\pi G \rho_s r_s^2 \\ \Phi(r) = 4\pi G \rho_s r_s^2 \ln(r/r_s) \end{array}}$$

How to calculate NFW profile for a halo with M_Δ at z ?

① concentration

$$c_\Delta = C_0(z) \left(\frac{M_\Delta}{10^{12} h^{-1} M_\odot} \right)^{-0.075} \left[1 + \left(\frac{M_{vir}}{M_0(z)} \right)^{0.26} \right]$$

where $C_0(z)$ & $M_0(z)$ is in a look-up table in Klypin+(2011)

② solve for r_Δ : $M_\Delta = \frac{4\pi}{3} \Delta_c \rho_c(z) \cdot r_\Delta^3$

where $\Delta_c(z) = 18\pi^2 + 82y - 39y^2$, $y \equiv \Omega_m(z) - 1$, $\Omega_m(z) = \frac{\Omega_{m,0}(1+z)^3}{\Omega_{m,0}(1+z)^3 + \Omega_{\Lambda,0}}$

$\simeq 178$ at $z > 4$, $\simeq 100$ at $z \sim 0$

for simplicity , $\Delta_c = 200$.

$$\rho_c(z) = \frac{3H^2(z)}{8\pi G} = 277\, M_\odot\, \mathrm{kpc}^{-3}\, h^2 \left[\Omega_{m,0}(1+z)^3 + \Omega_\Lambda \right]$$

③ $r_s = r_\Delta / c_\Delta$, ④ $\rho_s(M_\Delta, r_s, c_\Delta) \Rightarrow \rho_{NFW}(r; \rho_s, r_s)$

NFW rotation velocity curve

$$\left[\frac{V_c(r)}{V_\Delta}\right]^2 = \frac{1}{x}\frac{\ln(1+cx) - cx/(1+cx)}{\ln(1+c) - c/(1+c)}, \quad x \equiv r/r_\Delta \quad \text{(Navarro 98)}$$

where $V_\Delta = \sqrt{\dfrac{GM_\Delta}{r_\Delta}}$ virial velocity: the circular velocity at virial radius

$$= 207.4\,\text{km/s}\left[\frac{M_\Delta}{10^{12}M_\odot}\cdot\frac{100\,\text{kpc}}{R_\Delta}\right]^{1/2}$$

$$T_\Delta = \frac{1}{2}\mu m_p V_\Delta^2/k \quad \rightarrow \text{virial temperature}$$

$$= 3.6\times10^5\,\text{K}\left(\frac{V_\Delta}{100\,\text{km/s}}\right)^2 \quad \text{for } \mu = 0.59 \text{ (ionized H \& He)}$$

$$M_\Delta = 2.3\times10^5\,(V_\Delta/\text{km/s})^3\,h^{-1}M_\odot$$

$$M_\Delta \propto V_\Delta^3 \text{ because } V_\Delta \propto \sqrt{M_\Delta/r_\Delta},\ r_\Delta \propto M_\Delta^{1/3} \Rightarrow V_\Delta \propto M_\Delta^{1/3}$$

Maximum velocity

$$r_c^{max} = 2.16\,r_s = 2.16\,r_\Delta/c_\Delta$$

$$V_c^{max} = 1.64\,r_s\cdot\sqrt{G\rho_s}$$

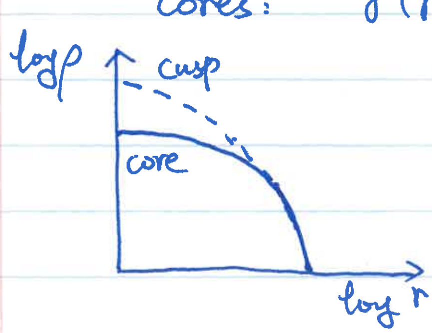
Logarithmic density slope (core vs. cusp)

$$\gamma(r) = \frac{d\ln\rho}{d\ln r} = \frac{r}{\rho}\frac{d\rho}{dr}, \quad \text{dimensionless}$$

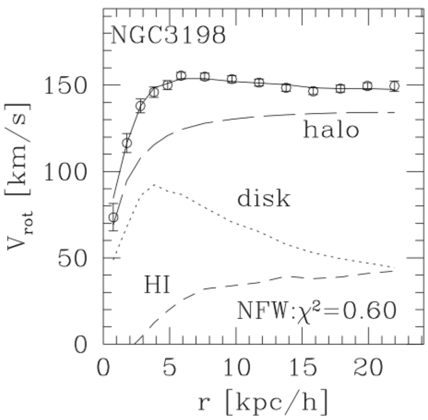
$$= -\frac{1+3r/r_s}{1+r/r_s} = \begin{cases} -1 & \text{at } r\to 0 \\ -2 & \text{at } r_s \\ -3 & \text{at } r\to\infty \end{cases}$$

density cusps: $\gamma(r\to 0) < 0$ (or -0.5) e.g. NFW & SIS.

cores: $\gamma(r\to 0) \simeq 0$ (or > -0.5)

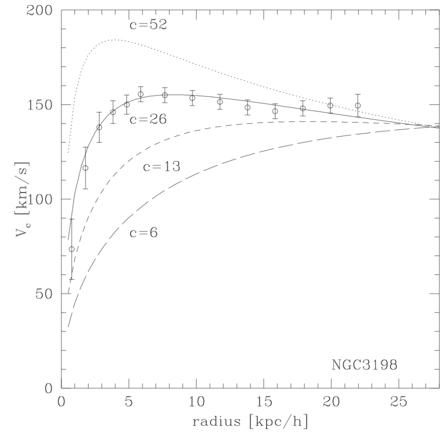


Fitting the Observed Rotation Curve of NGC 3198 w/ an NFW halo



Navarro 1998

NFW rotation curve's dependency on the concentration parameter



Navarro 1998