Boltzmann Eq. (1872): $\frac{n_i^k}{n_i^j} = \frac{g_i^k}{g_i^j} \exp[-(E_i^k - E_i^j)/kT] = \begin{cases} 0 & \text{when } T \to 0 \\ g_i^k/g_i^j & \text{when } T \to \infty \end{cases}$

Temperature at recombination: recombination completed at $z \sim 1100$

Saha Eq. (1920): $\frac{n_{i+1}}{n_i} = \frac{2 Z_{i+1}}{Z_i} \cdot \frac{1}{n_e} \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} \cdot e^{-\chi_i/kT}$

$$Z_i = \sum_{j=1}^{\infty} g_i^j \exp[-(E_i^j - E_i^1)/kT] \quad \text{(Partition Function)}$$

For pure H, $Z_{II} = 1$ (single proton), $Z_I = 2n^2 = 2$ (all at ground state)

$$\frac{n_p \cdot n_e}{n_{HI}} = \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-13.6\text{eV}/kT}, \quad \chi_i = 13.6\text{eV for Hydrogen}$$

$$\frac{n_p \cdot n_e}{n_{HI}} = \frac{(n_p/n_H)^2}{n_{HI}/n_H} \cdot n_H = \frac{f_{HII}^2}{1 - f_{HII}} \cdot n_H$$

where $n_H = n_{HI} + n_p$, $n_p = n_e$, $f_{HII} = n_p/n_H$ (ionization fraction)

given the total H density n_H, f_{HII} is a unique function of T.

Planck 2018 result: $\Omega_{b,0} = 0.02242\, h^{-2}$

$$\rho_{c,0} = \frac{3H_0^2}{8\pi G} = 1.8788 \times 10^{-26}\, h^2\, \text{kg}\, m^{-3}$$

$n_H : n_{He} = 12:1$

$\rho_H : \rho_{He} = 3:1$

(ignore Helium)

$$\Rightarrow \Omega_{b,0} \cdot \rho_{c,0} = \rho_{b,0} = 4.2 \times 10^{-28}\, \text{kg}/m^3$$

given $m_H = 1.673 \times 10^{-27}$ kg

$$\Rightarrow n_{H,0} = \rho_{b,0}/m_H = 0.25\, m^{-3}$$

for comparison, photons: $n_\gamma = 60\left(\frac{kT}{hc}\right)^3 = 4 \times 10^8 m^{-3}$ today

$kT = 8.6 \times 10^{-5}$ eV (T/K) $= 1.4 \times 10^{-23}$ J (T/K)

Redshift dependency:

$$n_H = n_{H,0} \cdot (1+z)^3 \quad ; \quad kT = 2.35 \times 10^{-4}\, \text{eV} \cdot (1+z) \; ; \quad T = 2.75\text{K}(1+z)$$

energy term on denominator:

$$\frac{h^2 n^{2/3}}{2\pi m_e} = 4.8 \times 10^{-18}\, \text{eV} \cdot (n/1\, m^{-3})^{2/3}$$

$m.f.p. = \frac{1}{n_e \sigma_e}$

@z=1100 = 370 kpc = 400 cMpc

$\sigma_e = 6.65 \times 10^{-29}\, m^2$

$n_e = n_{H,0}(1+z)^3 \cdot f_{HII}$

together, we have

$$\frac{f_{HII}^2}{1 - f_{HII}} = \left[\frac{2.35 \times 10^{-4}\, \text{eV}(1+z)}{1.9 \times 10^{-19}\, \text{eV} \cdot (1+z)^2}\right]^{3/2} \cdot \exp\left(-\frac{13.6\text{eV}}{2.35 \times 10^{-4}\, \text{eV}(1+z)}\right)$$

$$= 4.3 \times 10^{22} \cdot (1+z)^{-\frac{3}{2}} \cdot \exp\left[-\frac{57872}{(1+z)}\right]$$

At $z = 1100$, $f_{HII} = 0.004$

At $z = 1259$, $f_{HII} = 0.1$

At $z = 1377$, $f_{HII} = 0.5$

Horizon Distance in different eras: (Particle Horizon)

$k=0$ {
Comoving horizon distance: $\chi_h = c\cdot\int_0^t \frac{dt'}{a(t')}$
proper horizon distance: $d_h = a(t)\int_0^t \frac{c\,dt'}{a(t')} = a\cdot\chi_h = \frac{\chi_h}{1+z(t)}$

- radiation era ($z>3400$)
  $a(t)\propto t^{1/2} \Rightarrow d_h = 2ct \propto a^2,\ \chi_h\propto a$
- matter era ($z>0.3$)
  $a(t)\propto t^{2/3} \Rightarrow d_h = 3ct = \frac{2c}{H_0\sqrt{\Omega_{m,0}}}\cdot\frac{1}{(1+z)^{3/2}}$
  $d_h\propto a^{3/2},\ \chi_h\propto a^{1/2}$
- Λ era
  no simple analytical solution
  $d_h(t_0) = 14.6$ Gpc, as $t\to\infty$, $d_h\to 19.3$ Gpc

Epoches of recombination & matter-radiation equality:

$z\sim1100$, $t=370$ kyr, $d_h = 340$ kpc, $\chi_h = 374$ Mpc
$z\sim3400$, $t=51$ kyr, $d_h = 31$ kpc, $\chi_h = 105$ Mpc

mean. free path of photons at recombination:

def. optical depth: $\tau_\nu = \kappa_\nu \rho s = \sigma_\nu n s$

$m.f.p = s(\tau=1) = \frac{1}{\sigma_\nu n}$, $\sigma_{Thomson} = \frac{8\pi}{3}\left(\frac{q^2}{4\pi\varepsilon_0 mc^2}\right)^2 \propto q^4/m^2$

($h\nu \ll m_ec^2$) in early universe, Thomson scattering from e^- dominates cross-section.

$\sigma_e = 6.65\times10^{-29}\,m^2$ (freq. independent)
$n_e = n_{H,0}\cdot(1+z)^3\cdot f_{HII}$; $n_{H,0} = \frac{\Omega_{b,0}\rho_{c,0}}{m_H} = 0.25\,m^{-3}$
$m.f.p(z=1100) = 370$ pkpc for $f_{HII}=0.004$
$\simeq d_h(z=1100) = 340$ kpc.

Thomson scattering condition: $h\nu \ll m_ec^2 = 511$ ~~kpc~~ keV (hard X-ray)
Compton scattering condition: $h\nu \gg m_ec^2$

Sonic horizon distance & the first peak in CMB power spectrum

proper particle horizon: $d_h = a(t)\int_0^t \frac{dt'}{a(t')} = a\,x_h$

before recombination: $c_s = c/\sqrt{3}$, radiation pressure dominated

sonic horizon distance $d_s = d_h/\sqrt{3}$

because $t_{eq} \ll t_{recomb}$, most of the growth in horizon is in the matter era (EdS):

$$d_h = 3ct \Rightarrow d_s = \sqrt{3}\,ct$$

to calculate the angular size of d_s, we need the angular diameter distance:

$$d_A = d_p/(1+z) = \frac{1}{(1+z)}\cdot a(t_0)\cdot\int_{t_{rec}}^{t_0}\frac{c\cdot dt}{a(t)}$$

$$a(t) = \left(\frac{3}{2}\right)^{2/3}\cdot\left(\frac{t}{t_H}\right)^{2/3} \text{ for EdS, where } t_H = \frac{3}{2}t_0$$

$$\Rightarrow d_A = \frac{1}{1+z_{rec}}\cdot 3ct_0\left[1-\left(\frac{t_{rec}}{t_0}\right)^{1/3}\right]$$

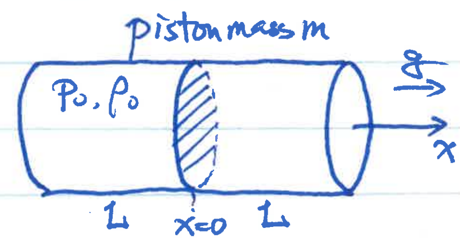
the angular size of the horizon is then

$$\theta = \frac{d_s(t_{rec})}{d_A(t_{rec})} = \frac{200\text{ kpc}}{12.4\text{ Mpc}} = 0.93^\circ \text{ for } t_{rec}=380\text{ kyr},\ z_{rec}=110$$

$$l = \frac{\pi}{\theta} = \frac{180^\circ}{0.93^\circ} = 194 \simeq l_{obs} = 200$$

Note that
$\theta = d_s/d_A$
$= x_s/d_p$
$= x_s/x$
comoving dis. ratio

Acoustic Oscillations – a simple cylinder model



Newton's 2nd law:

$$m\ddot{x} = A\cdot\Delta P$$

$$\Delta P = -2c_s^2\rho_0\left(\frac{x}{L}\right) \text{ because } \Delta P = \frac{dP}{d\rho}(\Delta\rho_1-\Delta\rho_2)$$

$$m = 2LA\rho_0$$

$$\Rightarrow \ddot{x} = -\frac{c_s^2}{L^2}\cdot x \Rightarrow x = x_0\sin(\omega t),\ \boxed{\omega = c_s/L,\ \text{Period} = 2\pi/\omega}$$

$$\text{Period} = \frac{2\pi L}{c_s},\quad c_s = \frac{c}{\sqrt{3}} \text{ before decoupling}$$

Spherical Collapse of baryonic matter: Jeans minimum mass criteria

Ideal Gas.

$K \propto R^3$
$U \propto R^5$

① $2K < |U|$

$$\frac{3MkT}{\mu m_H} < \frac{3}{5}\frac{GM^2}{R}, \quad \mu \equiv \frac{\langle m \rangle}{m_H}$$

(average mass per particle → $\langle m \rangle$; mean molecular mass → μ)

$$R = \left(\frac{3M}{4\pi\rho_0}\right)^{1/3}$$

$$\Rightarrow M > M_J = \left(\frac{5kT}{G\mu m_H}\right)^{3/2}\left(\frac{3}{4\pi\rho_0}\right)^{1/2}$$

$$R > R_J = \left(\frac{15kT}{4\pi G\mu m_H \rho_0}\right)^{1/2}$$

② $t_{ff} < r_0 / c_s$

$$t_{ff} = \left(\frac{3\pi}{32 G\rho_0}\right)^{1/2}$$

adiabatic

$$c_s = \sqrt{\frac{\partial P}{\partial \rho}} = \sqrt{\frac{\gamma kT}{\mu m_H}}, \quad P \propto \rho^\gamma$$

$$\Rightarrow R > R_J = \left(\frac{3\pi\gamma kT}{32 G\mu m_H \rho_0}\right)^{1/2}$$

$$M > M_J = \left(\frac{\gamma kT}{G\mu m_H}\right)^{3/2} \cdot \left(\frac{3\pi^4}{2^{11}}\frac{1}{\rho_0}\right)^{1/2}$$

The two criteria give the same results when $\gamma = \gamma_c = 40/\pi^2 \sim 4$

γ_c is higher than the values for monatomic & diatomic gas: $\gamma = \frac{5}{3}, \frac{7}{5}$

so ② gives $1.5\times$ smaller R_J & $3.7\times$ smaller M_J than ①.

$M_J \propto T^{3/2}\rho^{-1/2}$ it decreases as $T \searrow$ & $\rho \nearrow$

the coldest & densest regions in a nebula collapses first.

Jeans (1902) The stability of a Spherical Nebula

Formal derivation based on perturbation analysis gives (§8.3.2 CFN20)

$$\lambda_J = \sqrt{\frac{\pi}{G\rho}} \cdot c_s = c_s \cdot t_{ff} \cdot \sqrt{\frac{32}{3}} \simeq 3.27\, c_s \cdot t_{ff}$$

Poisson Eq. of density perturbation $\delta\rho$:

$$\frac{\partial^2 \delta\rho}{\partial t^2} = c_s^2 \nabla^2 \delta\rho + 4\pi G\rho_0 \delta\rho$$

plug in $\delta\rho = \hat{\delta\rho}\, e^{i(\vec{k}\cdot\vec{x} - \omega t)}$ we get the dispersion relation:

$$\omega^2 = c_s^2 k^2 - 4\pi G\rho_0 = c_s^2\left(\frac{2\pi}{\lambda}\right)^2 - 4\pi G\rho_0\ ; \quad \text{instability } \omega^2 < 0 \Rightarrow \lambda_J$$

Jeans length in Expanding Universe (Photon-Baryon Fluid)

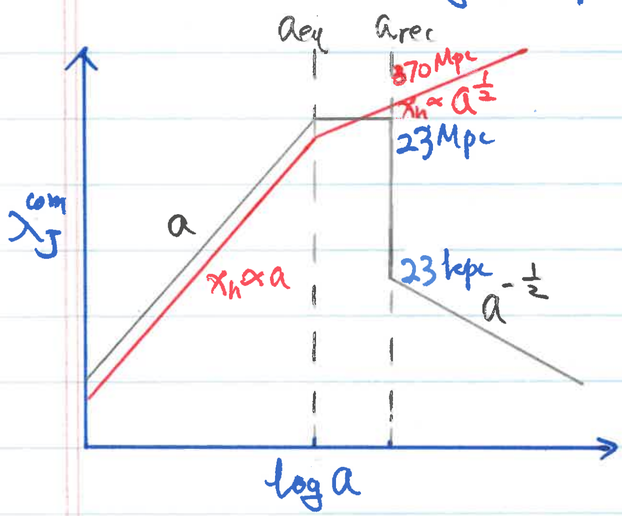
$$\lambda_J^{prop} \simeq c_s \sqrt{\frac{\pi}{G\rho}} \propto c_s \cdot \rho^{-\frac{1}{2}}$$ here ρ is total density (matter + radiation)

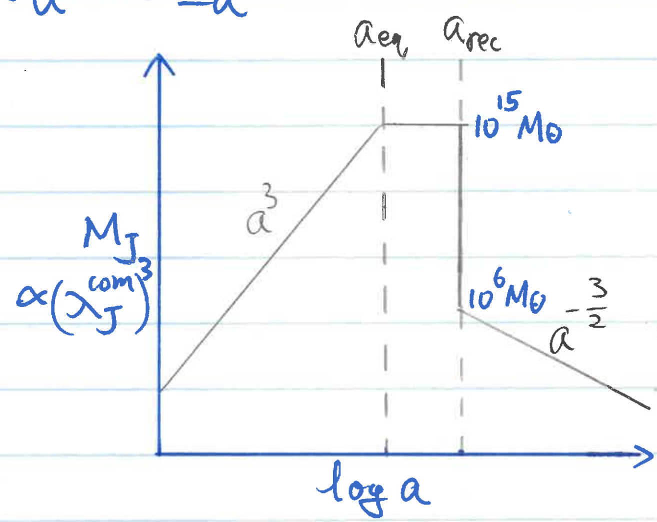
$t < t_{eq} \Rightarrow \begin{cases} \rho = \rho_{rad} \propto a^{-4} \\ c_s = c/\sqrt{3} \end{cases} \Rightarrow \lambda_J^{prop} \propto a^2 \Rightarrow \lambda_J^{com} = \frac{\lambda_J^{prop}}{a} \propto a$

$t_{eq} < t < t_{rec} \Rightarrow \begin{cases} \rho = \rho_m \propto a^{-3} \\ c_s = \frac{c}{\sqrt{3}}\left[\frac{3}{4}\frac{\rho_b}{\rho_r} + 1\right]^{-\frac{1}{2}} \propto a^{-\frac{1}{2}} \end{cases} \Rightarrow \lambda_J^{prop} \propto a \;\&\; \lambda_J^{com} \propto a^0$

$t > t_{rec} \Rightarrow \begin{cases} \rho = \rho_m \propto a^{-3} \\ c_s = \left(\frac{\gamma k T_{gas}}{\mu m_H}\right)^{1/2} \propto a^{-1} \end{cases} \Rightarrow \lambda_J^{prop} \propto a^{\frac{1}{2}} \;\&\; \lambda_J^{com} \propto a^{-\frac{1}{2}}$

where $T_{gas} \propto V^{1-\gamma} \propto a^{3\cdot(-\frac{2}{3})} = a^{-2}$





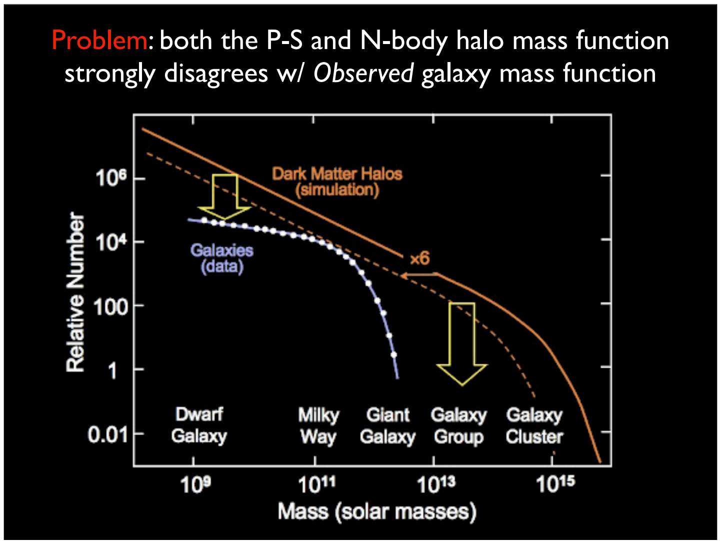
At z_{rec}, $c_s = 5.8$ km/s, $\bar{\rho}_m = 3.7\times10^{-21}$ g/cm³, $\lambda_J^{prop} = 21$ pc, $M_J = 2\times10^6 M_\odot$

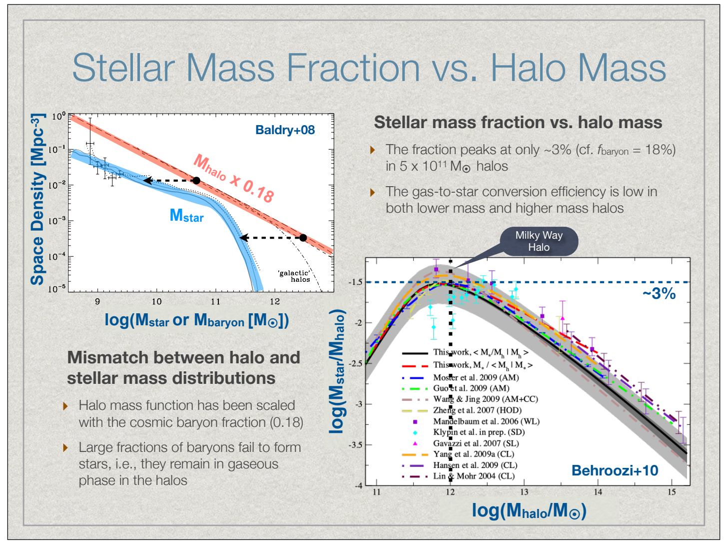
Comoving Particle Horizon

$$x_h = \int_0^t \frac{c\,dt'}{a(t')} = \int_0^a \frac{c\,da}{a^2(\dot{a}/a)} = \sqrt{\frac{3c^2}{8\pi G}} \int_0^a \frac{da}{a^2\sqrt{\rho}}$$

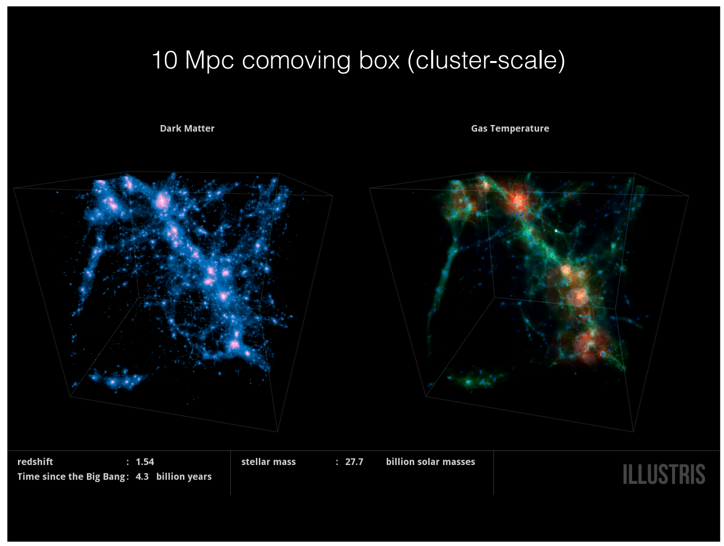
where we have used F.E. for $k=0$: $\frac{\dot{a}}{a} = \sqrt{\frac{8\pi G}{3}\rho - \frac{kc^2}{a}}$

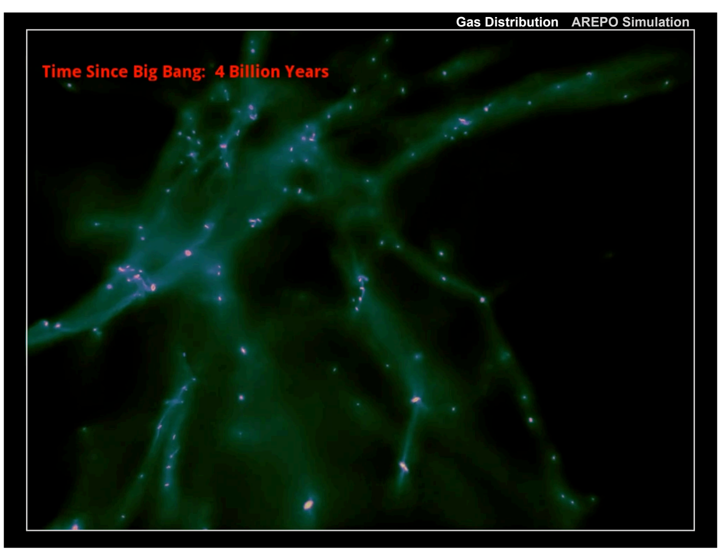
$t < t_{eq}$; $\rho \propto a^{-4} \Rightarrow x_h \propto a$; at $t > t_{eq}$, $\rho \propto a^{-3} \Rightarrow x_h \propto \int a^{-\frac{1}{2}} da \propto a^{\frac{1}{2}}$

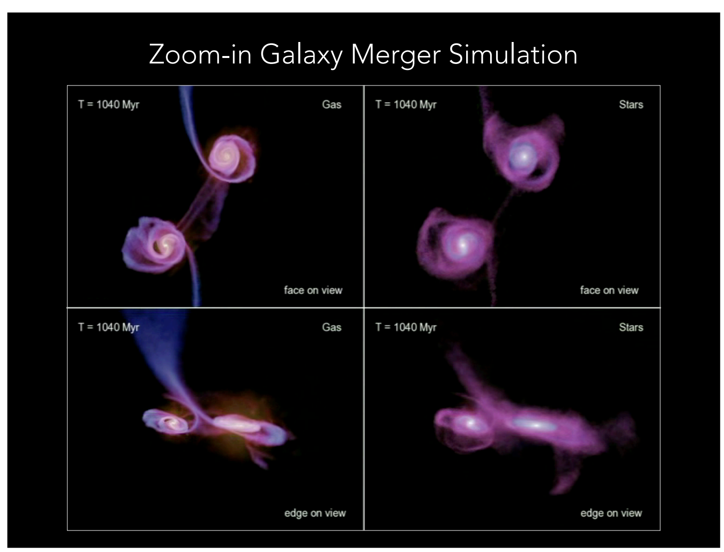


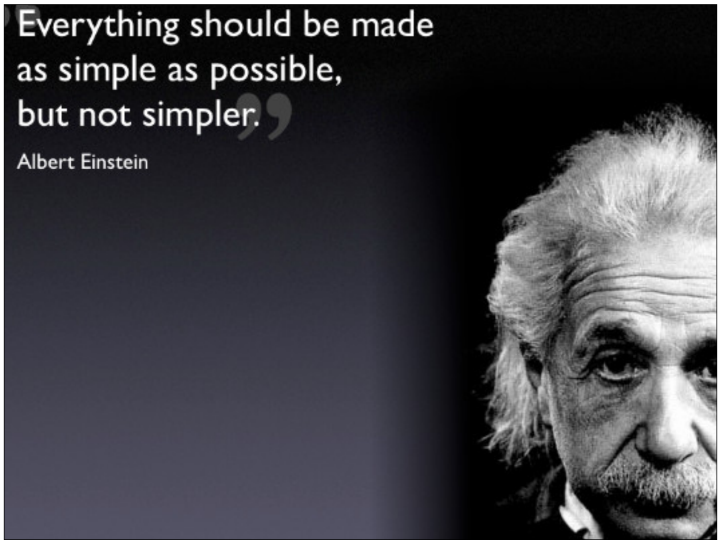


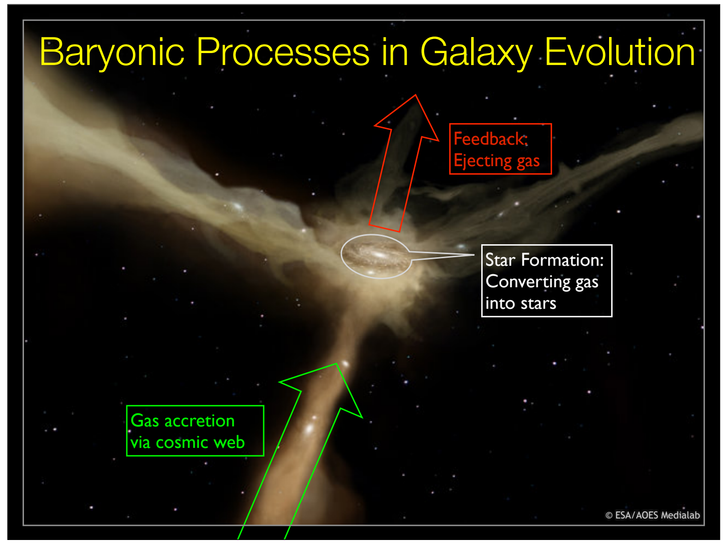
Cosmological N-body plus hydrodynamical simulations: DM + baryons

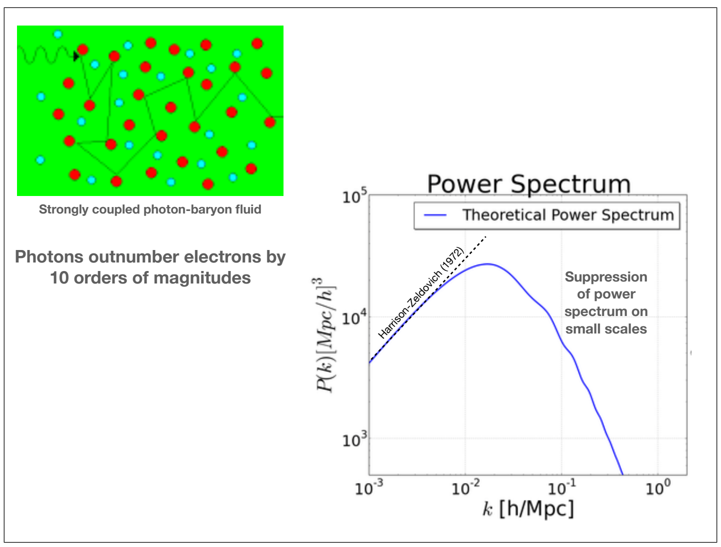








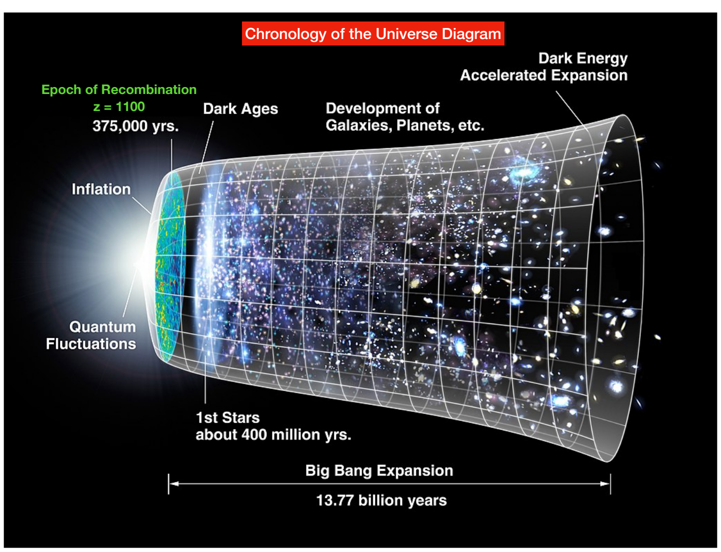


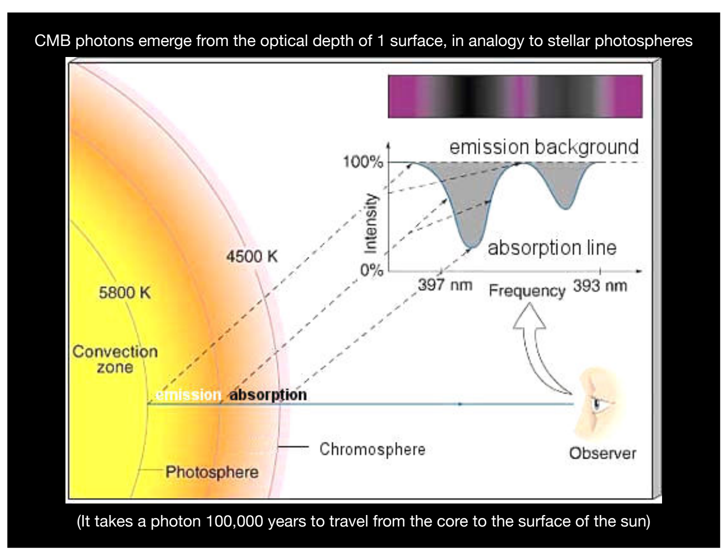


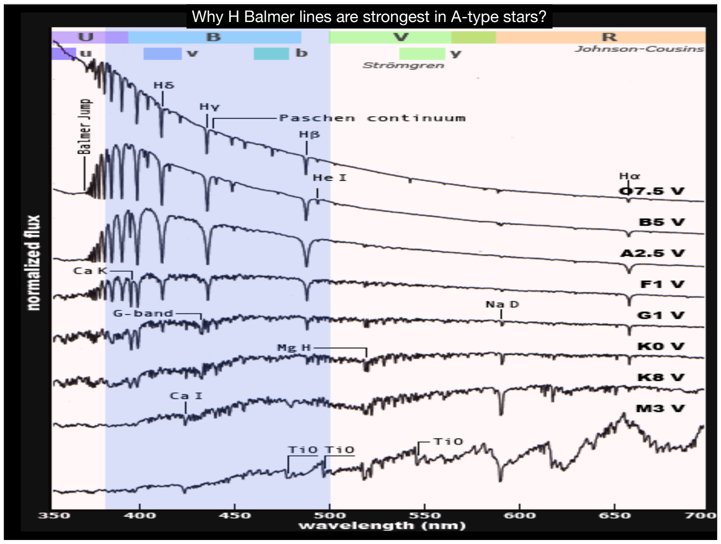
So before baryons can gravitationally collapse and form the first stars and galaxies, they must decouple from the photons

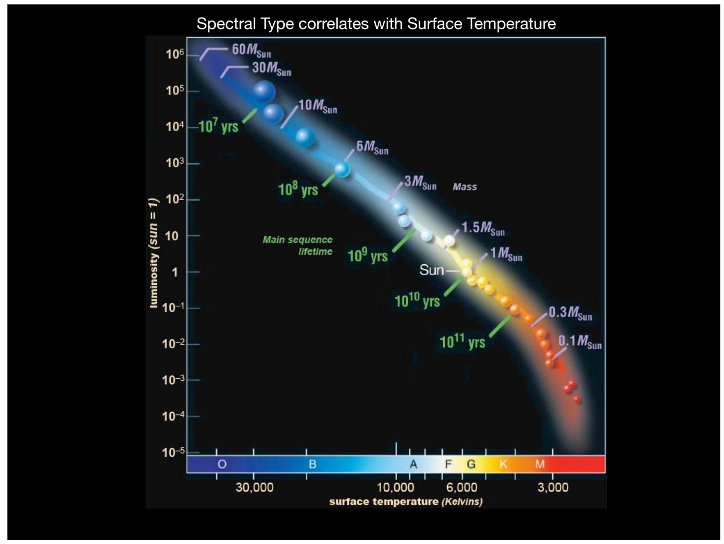
$$p^+ + e^- \rightarrow H + \gamma$$

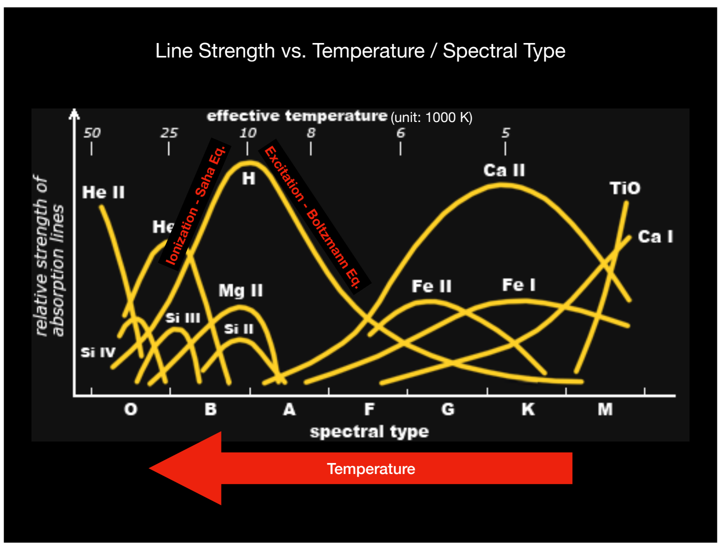
The decoupling epoch marks both the recombination of Hydrogen and the last scatter of CMB photons



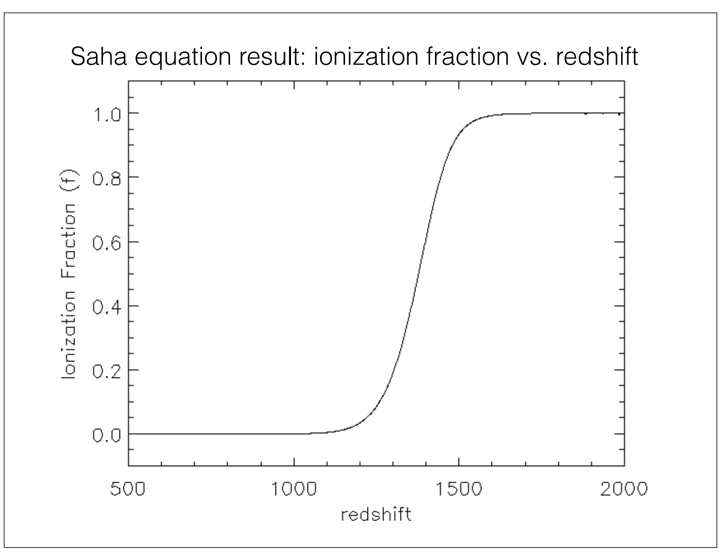


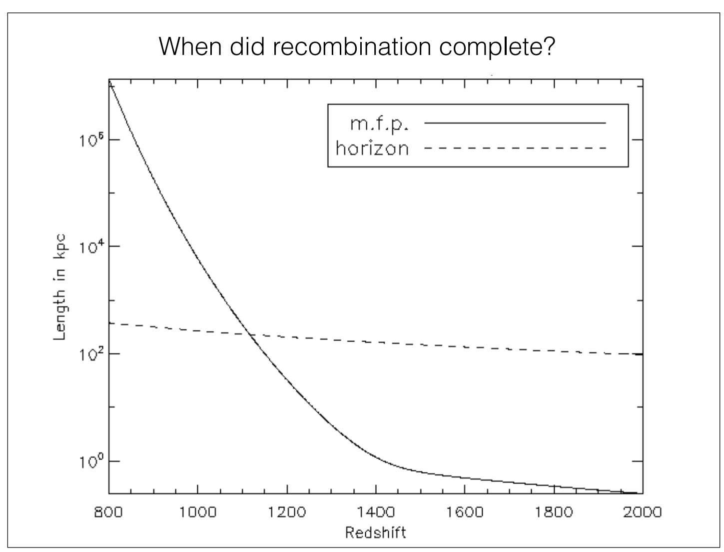






Apply Saha (1920) Ionization Eq. to the Universe to calculate the ionization fraction vs. redshift





How recombination affects the photons?

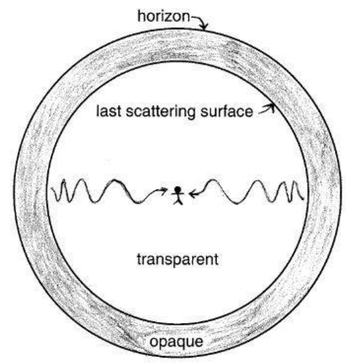
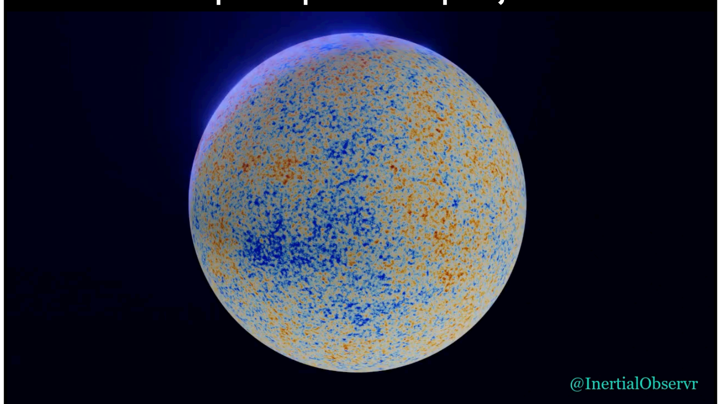


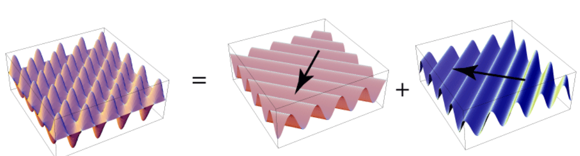
Figure 9.3: An observer is surrounded by a spherical last scattering surface. The photons of the CMB travel straight to us from the last scattering surface, being continuously redshifted.

δT map in spherical projection



Quantifying CMB anisotropies w/ temperature power spectrum

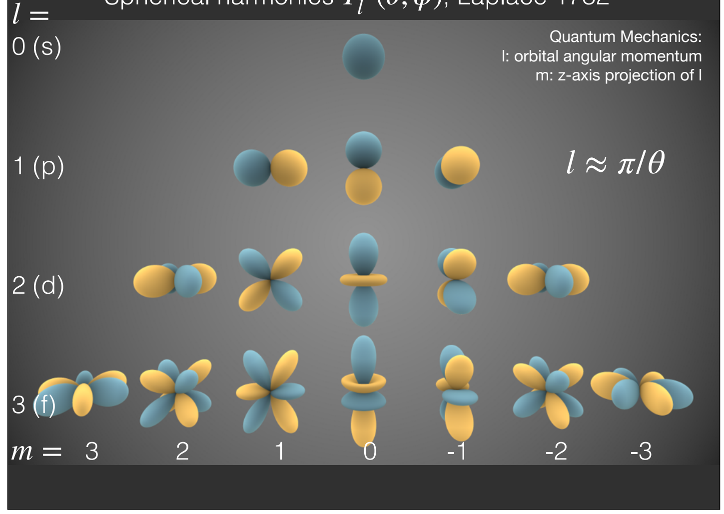
Analogy: Expressing periodic density fluctuations as the sum of Fourier bases



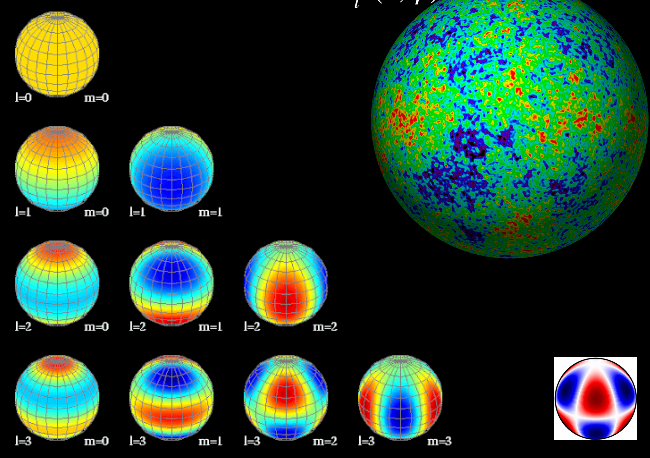
Fourier Transform: $\delta(\boldsymbol{x}) = \sum_{k=2\pi/l}^{\infty} \delta(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$

Power Spectrum: $P(k) = V\langle |\delta(\boldsymbol{k})|^2 \rangle$

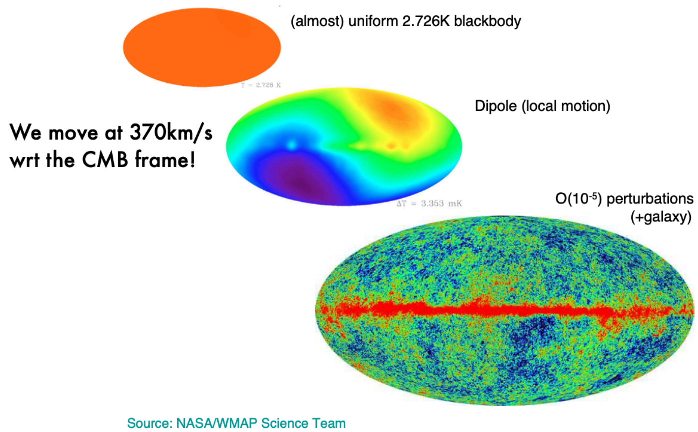
Spherical harmonics $Y_l^m(\theta, \phi)$, Laplace 1782



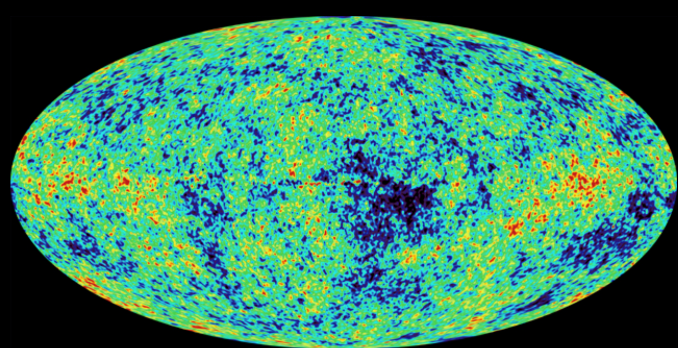
Representing CMB anisotropies as a sum of spherical harmonics $Y_l^m(\theta, \phi)$



CMB - Scaler - Dipole = MW + Anisotropies

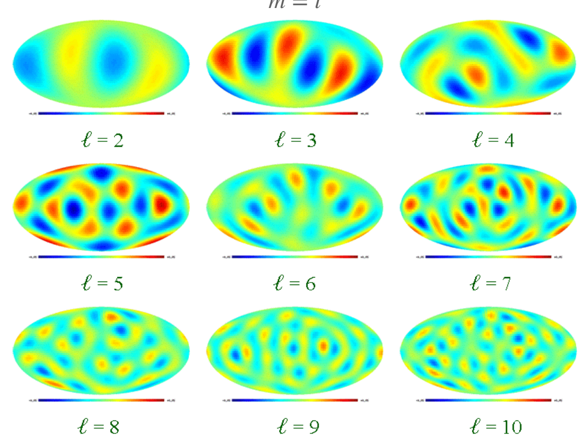


Mollweide (equal-area) projection

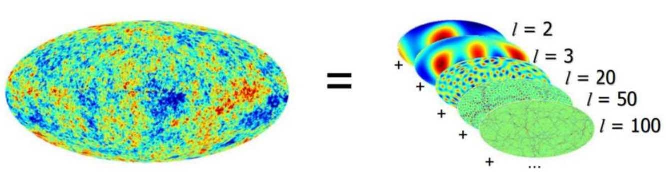


Spherical harmonics in Mollweide projection

$m = l$



Expressing anisotropies as sum of spherical harmonics



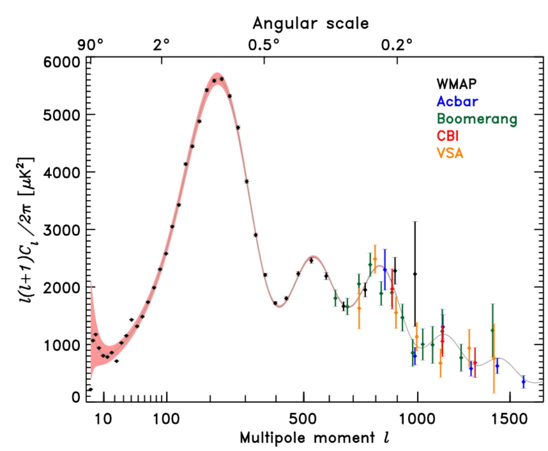
Harmonic Decomposition:

$$\delta_T(\theta, \phi) = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} a_{l,m} Y_l^m(\theta, \phi)$$

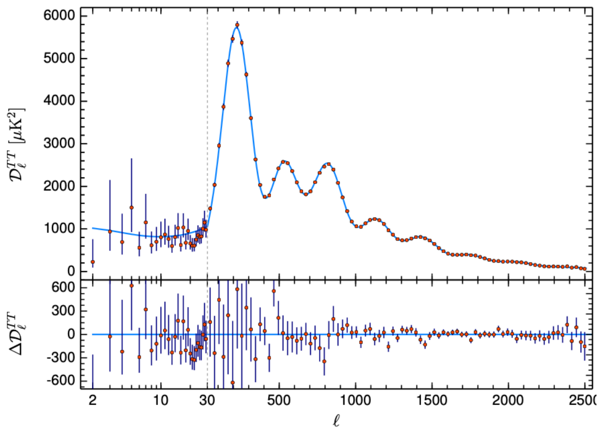
Temp. Power Spectrum:

$$P(l) = \frac{l(l+1)}{2\pi} C_l = \frac{l(l+1)}{2\pi} \frac{1}{2l+1} \sum_{m=-l}^{l} |a_{l,m}|^2$$

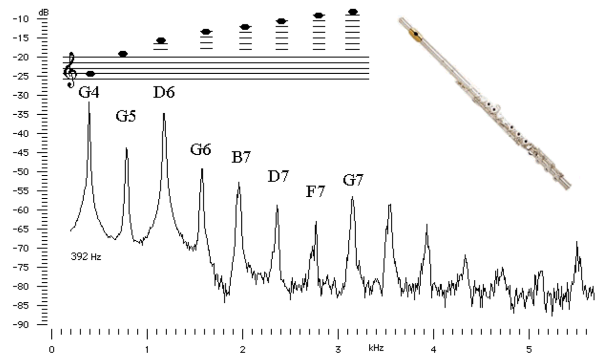
WMAP temperature power spectrum



Planck temperature power spectrum



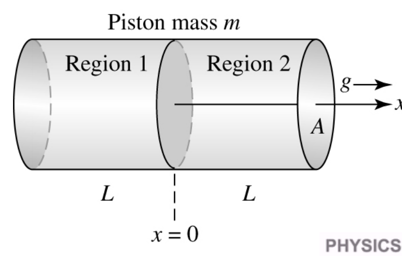
Harmonic spectrum of a flute



https://www.intmath.com/fourier-series/6-line-spectrum.php

Explaining the peaks in the temperature power spectrum

Because overdensities of baryon+photon fluid cannot collapse (Jeans length > Horizon), they under go acoustic oscillations



Ideal Gas Solution

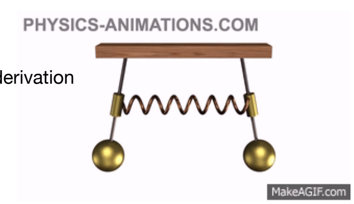
$$x(t) = x_0 \sin(c_s t/L)$$

$$\tau = 2\pi L/c_s$$

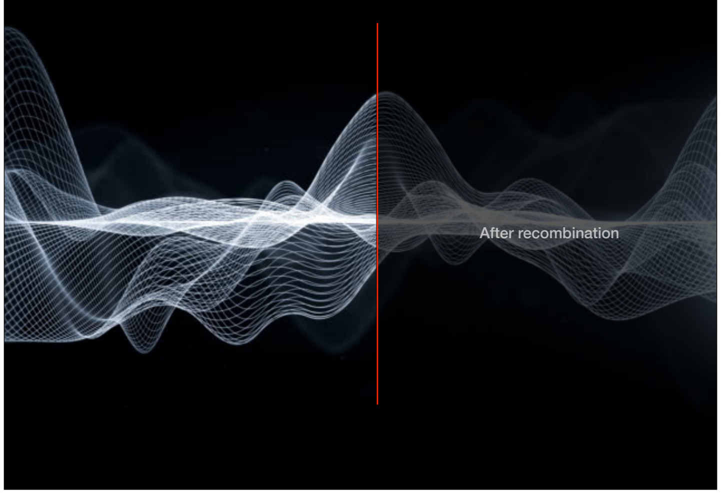
$$c_s^2 = \frac{\partial P}{\partial \rho} = \frac{\gamma k T}{\mu m_H}$$

$$\gamma = C_P/C_V$$

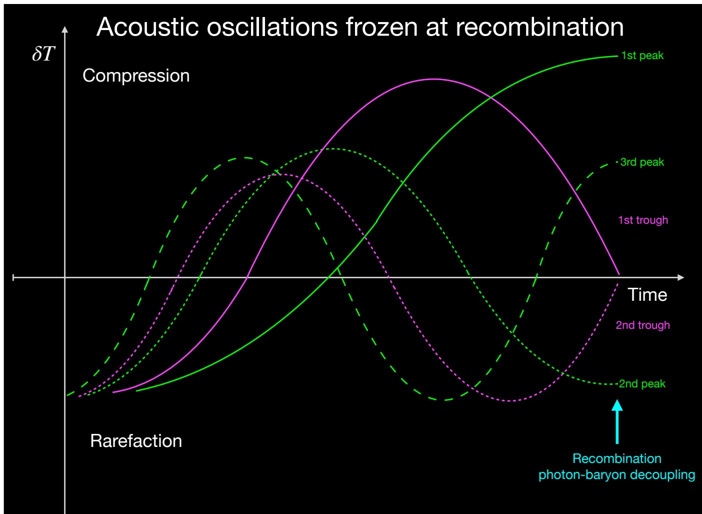
Simple gas cylinder + piston model derivation



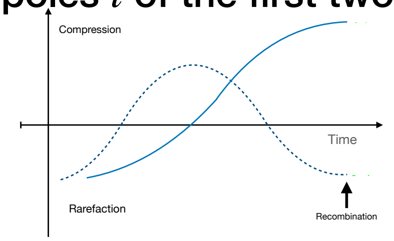
random acoustic oscillations frozen at recombination



Acoustic oscillations frozen at recombination

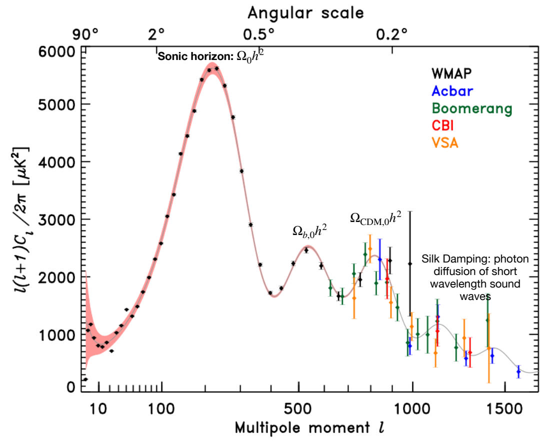


Multipoles l of the first two peaks

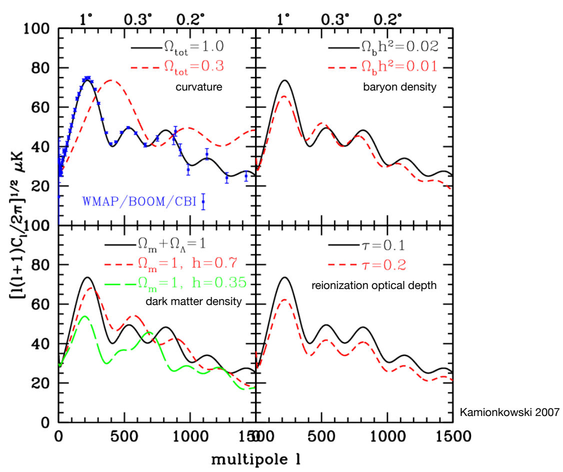


- First peak ($l \sim 200$): the largest structures that could have reached maximum compression at recombination: $\tau = 2\pi L/c_s = 2t_{\rm rec} \rightarrow L \sim c_s t_{\rm rec} \sim$ sonic horizon
- Second peak ($l \sim 500$): the largest structures that could have reached maximum rarefaction $\tau = 2\pi L/c_s = t_{\rm rec} \rightarrow L \sim c_s t_{\rm rec}/2 \sim$ sonic horizon/2

CMB power spectrum & cosmological parameters



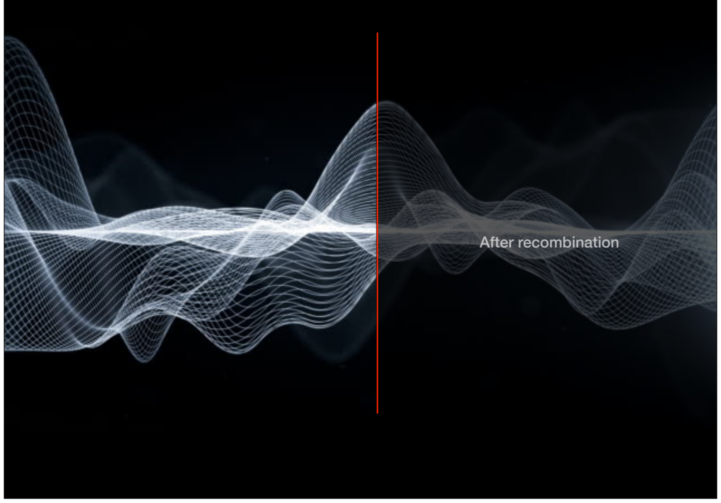
CMB power spectrum & cosmological parameters



Kamionkowski 2007

Phase Coherence Problem:
who is the conductor?

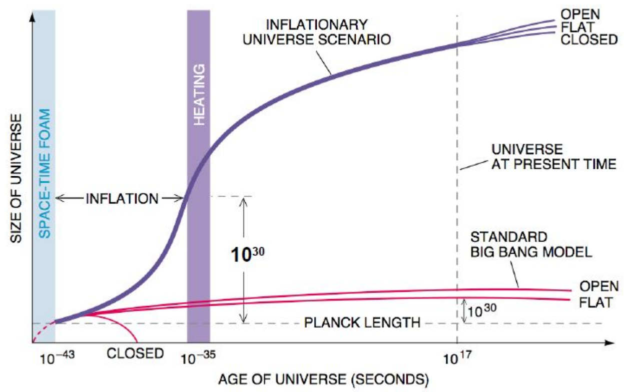
random acoustic oscillations frozen at recombination

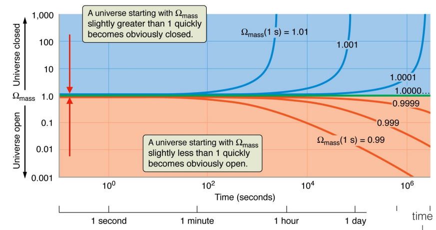


Implications of the existence of strong harmonic peaks

- Oscillations of all density fluctuations of a given size (thus having the same frequency) must reach their maximum compressions / rarefactions at the same time.
- This requires that they begin their oscillations simultaneously and with coherent phases
- In other words, to play the cosmic symphony the universe needs a conductor

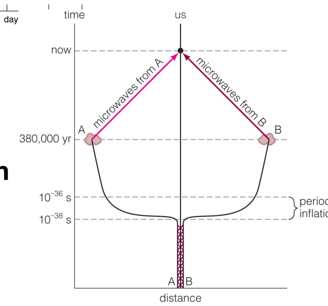
The inflation theory



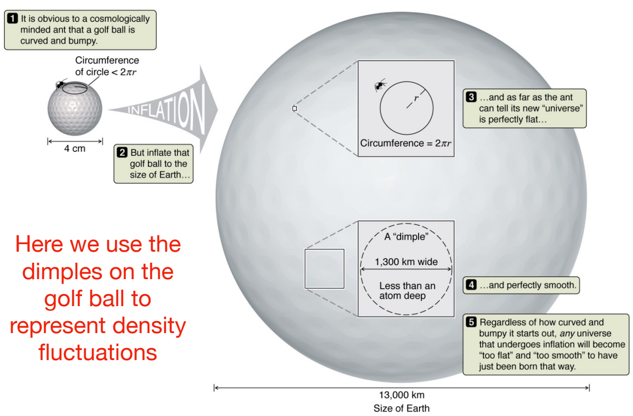


Flatness Problem

Horizon Problem

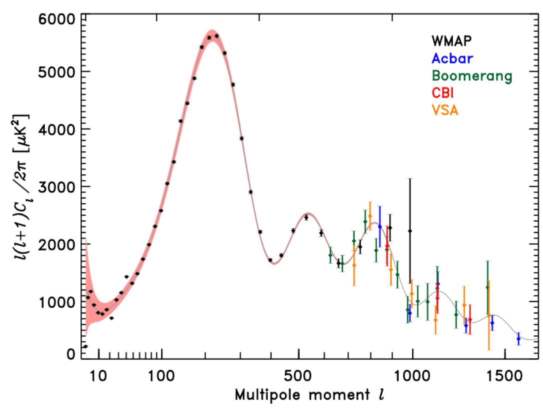


Inflation also solves the phase coherence problem by expanding density fluctuations to super-horizon sizes



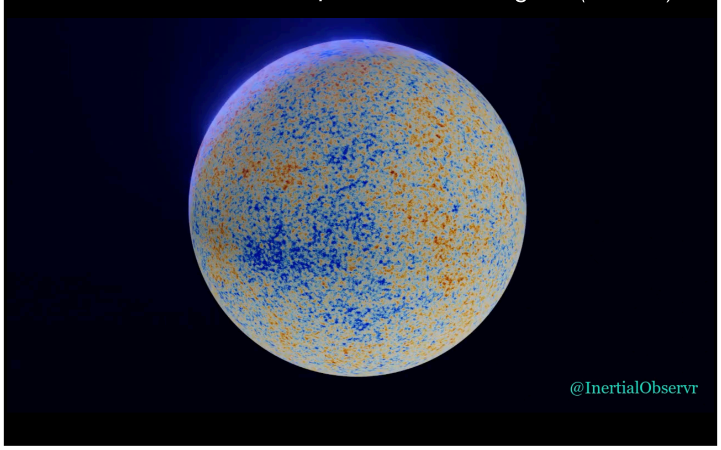
Here we use the dimples on the golf ball to represent density fluctuations

Inflation made sure that oscillations of all density fluctuations of a given size reach their maximum compressions / rarefactions at the same time, leading to the strong peaks in the power spectrum



The cosmic harmonics frozen in time

"What makes the music of heaven?" - Chuang Tzu (300 BC)



How recombination changes the baryons?