$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho(\vec{r}')}{\Delta r} d\tau'$$

2-d case:

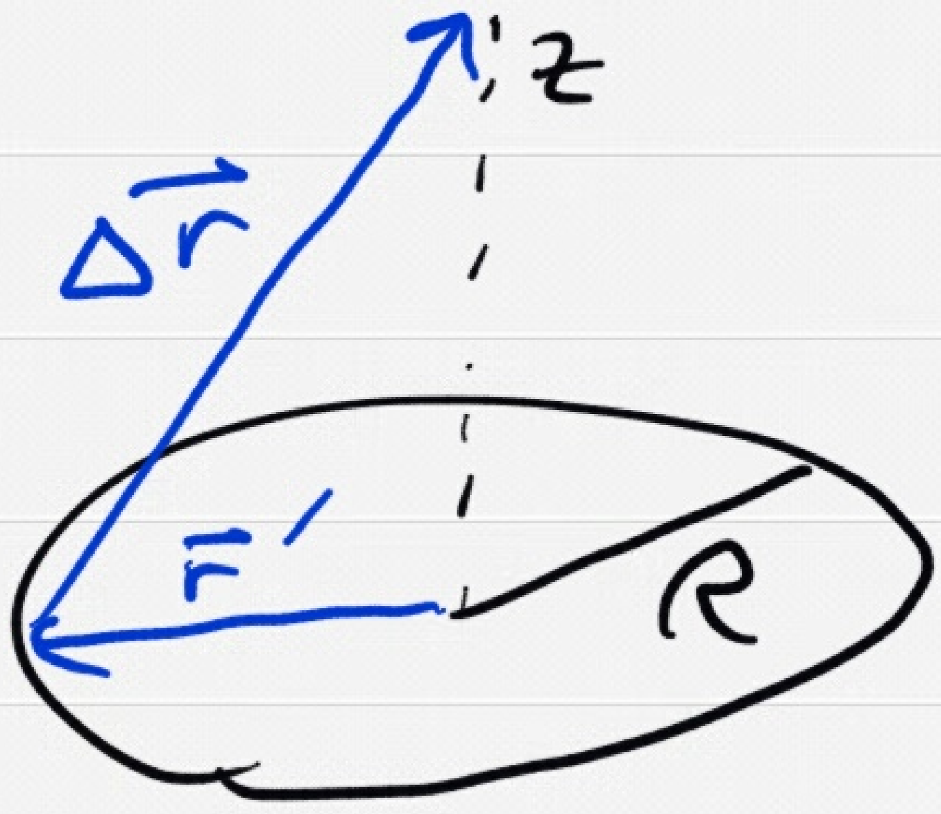
$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_A \frac{\sigma(\vec{r}')}{\Delta r} da'$$

1-d case

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_L \frac{\lambda(\vec{r}')}{\Delta r} dl'$$

All assume $V(\infty) = 0$

Example: Charged Ring



V on z-axis

$\lambda = Q/2\pi R$

$$|\Delta\vec{r}| = \sqrt{R^2 + z^2}$$

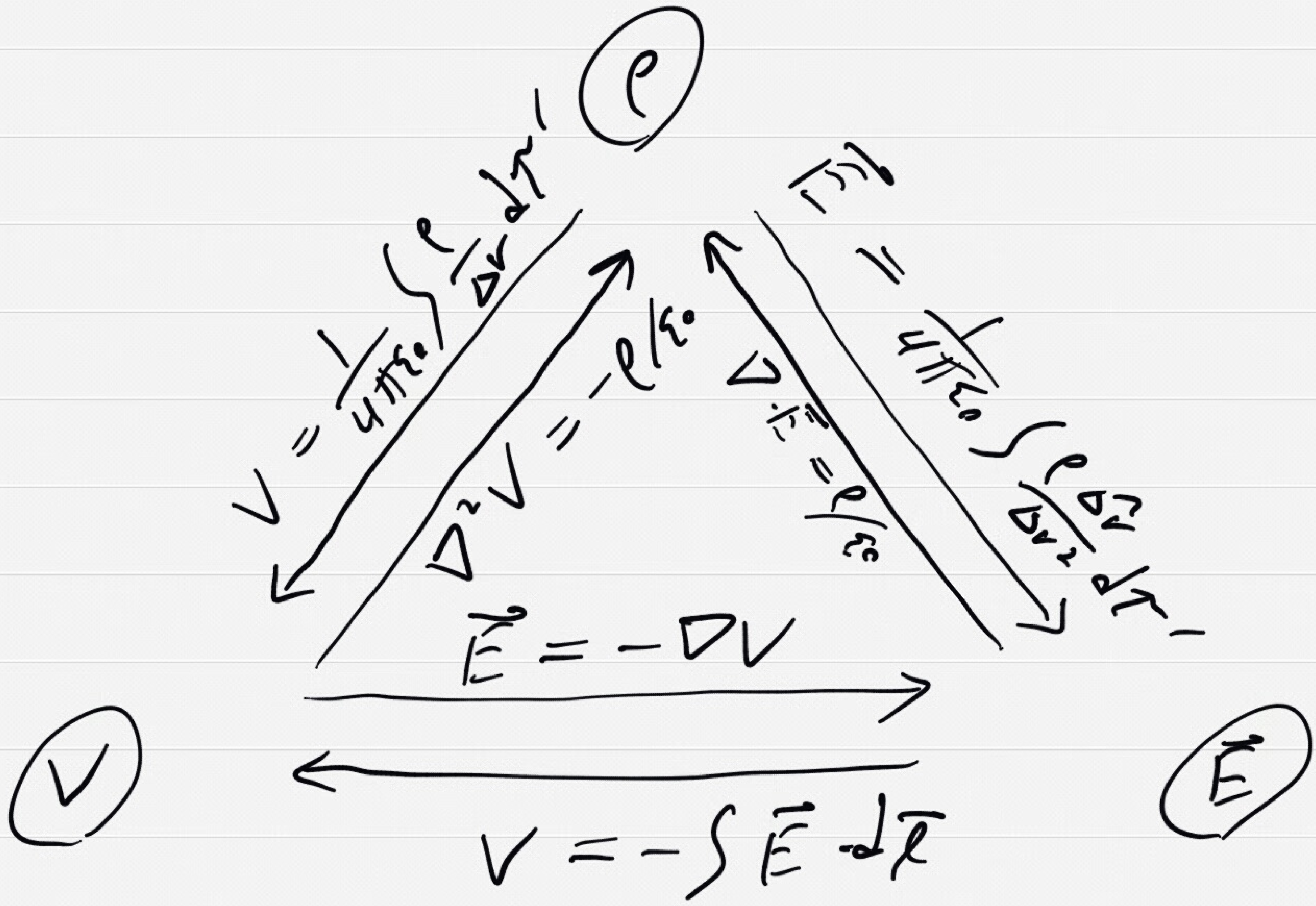
$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_L \frac{\lambda}{\sqrt{R^2+z^2}} dl'$$

$$= \boxed{\frac{1}{4\pi\varepsilon_0} \frac{\lambda \cdot 2\pi R}{\sqrt{R^2+z^2}}} = \boxed{\frac{Q}{4\pi\varepsilon_0} \cdot \frac{1}{\sqrt{R^2+z^2}}}$$

$$\vec{E}_z = -\partial V/\partial z$$

$$= -\frac{\lambda \cdot 2\pi R}{4\pi\varepsilon_0} \cdot \frac{-1/2 \cdot 2z}{(R^2+z^2)^{3/2}} = \boxed{\frac{\lambda \cdot 2\pi R}{4\pi\varepsilon_0} \cdot \frac{z}{(z^2+R^2)^{3/2}}}$$

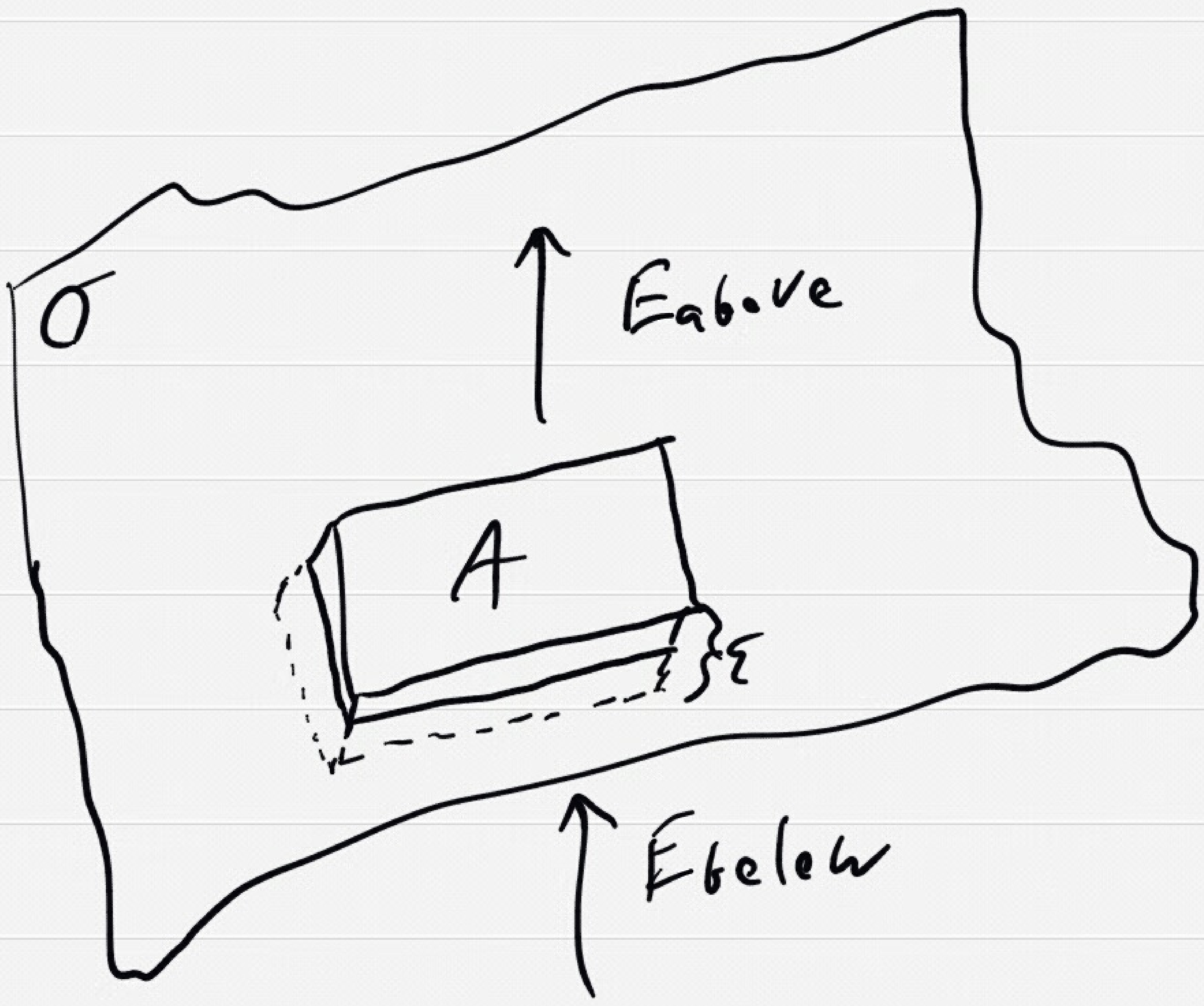
$\boxed{\vec{E},\ \rho,\ V}$



- Problem dictates which is most useful

- Sometimes $\rho \to V \to \vec{E}$ easier than $\rho \to \vec{E}$ etc.

Boundary Conditions



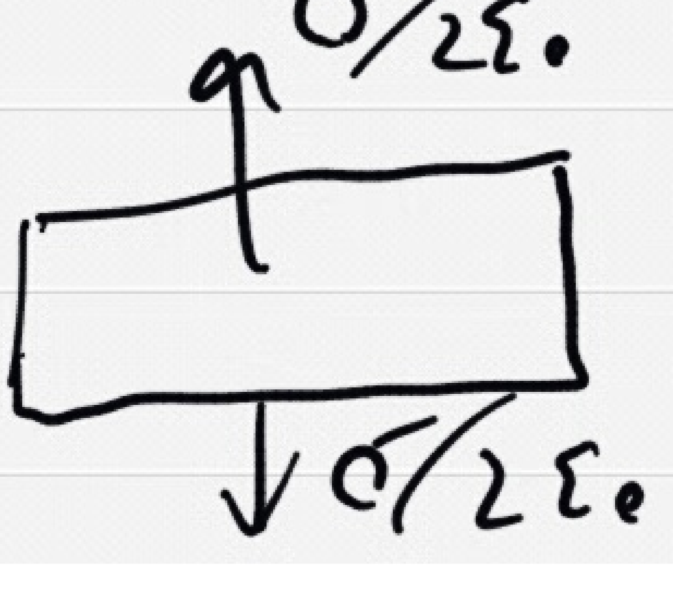
- Gaussian pillbox, area A, thickness ε

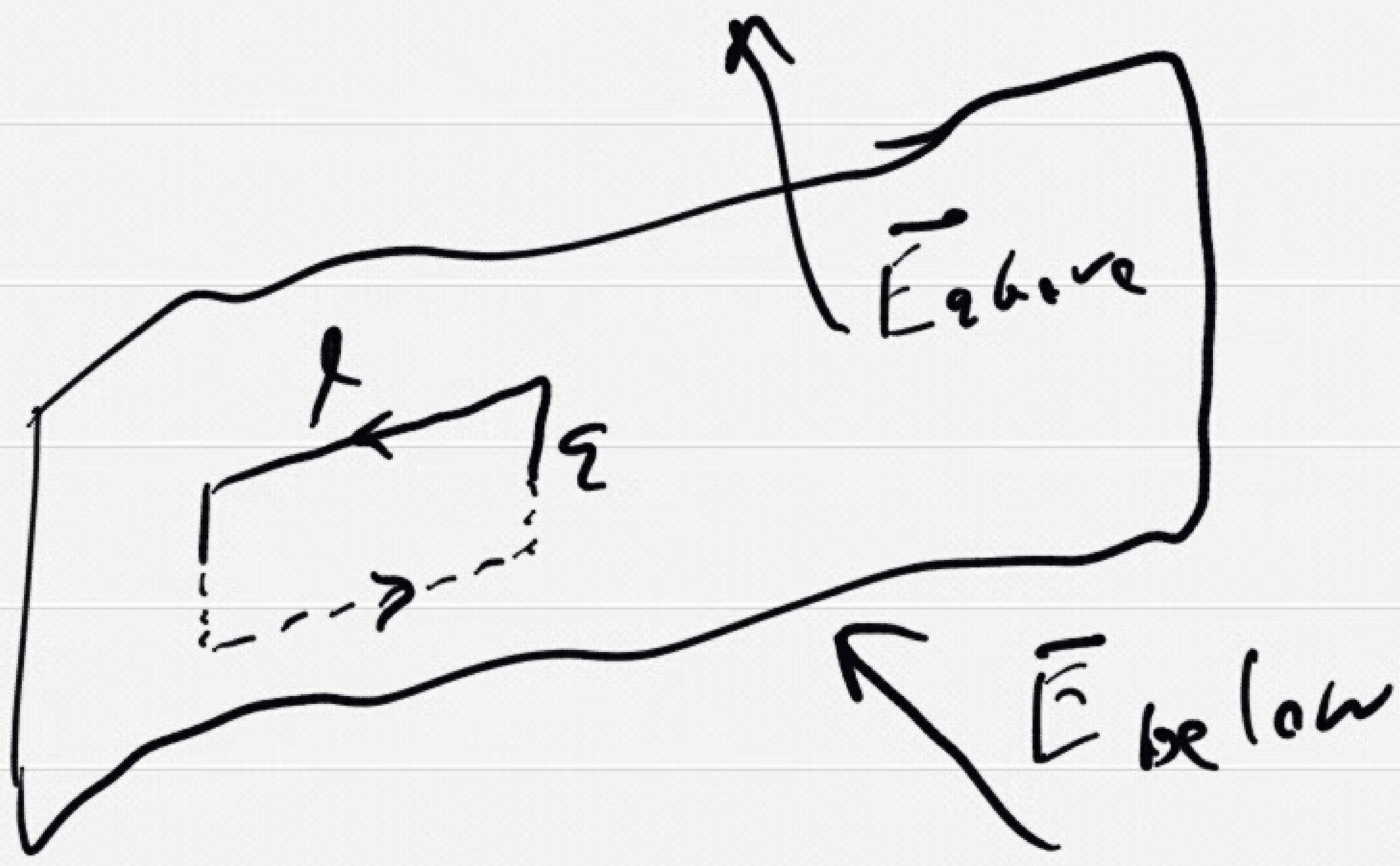
- Let $\varepsilon \to 0$, still containing surface

$$\oint \vec{E} \cdot d\vec{a} \to E_{\perp\,above} \cdot A - E_{\perp\,below} \cdot A$$
$$= \Delta E_\perp \cdot A$$
$$= Q_{enc}/\varepsilon_0 = \sigma A/\varepsilon_0$$

$$\Rightarrow \boxed{\Delta E_\perp = \sigma/\varepsilon_0}$$

Plane of charge is special case





- Gaussian loop of length l
- Again, let $\varepsilon \to 0$

$$\oint \vec{E} \cdot d\vec{l} \to \vec{E}_{above} \cdot \vec{l} - \vec{E}_{below} \cdot \vec{l}$$
$$= \Delta\vec{E} \cdot \vec{l}$$
$$= 0 \quad \text{for any } \vec{l} \parallel \text{ to surface}$$
$$\Rightarrow \boxed{\Delta \vec{E}_{\parallel} = 0}$$

Combine: $\Delta E_{\perp} = \sigma / \varepsilon_0$

$\Delta \vec{E}_{\parallel} = 0$

$$\Rightarrow \boxed{\Delta \vec{E} = \sigma / \varepsilon_0 \, \hat{n}}$$

w/ $\hat{n}$ from "below" to "above"

Note: V always continuous if $\vec{E}$ finite

Work

- Work done on charge Q moving from $\bar{a}$ to $\vec{b}$

$$W_E = \int_{\bar{a}}^{\vec{b}} \bar{F} \cdot d\bar{l} \quad \text{done by Coulomb force}$$

$$= \int_{\vec{a}}^{\vec{b}} Q \vec{E} \cdot d\vec{l}$$

Work done to oppose Coulomb force

$$W_{on} = -W_E$$

$$= -\int_{\vec{a}}^{\vec{b}} Q \vec{E} \cdot d\vec{l}$$

$$= Q \left[V(\vec{b}) - V(\vec{a}) \right]$$

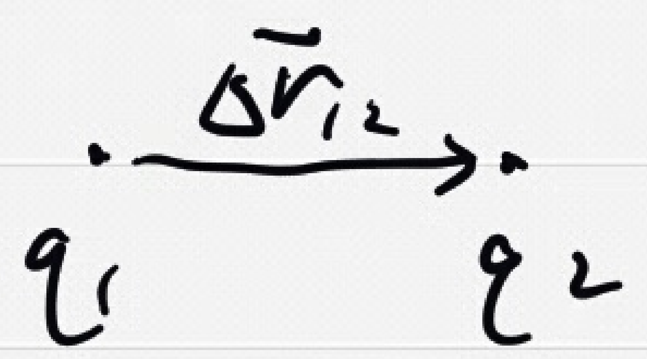
$$\Rightarrow W_{on}/Q = V(\vec{b}) - V(\vec{a})$$

if $\bar{a} = \infty$, $W_{on} = QV(\vec{r})$

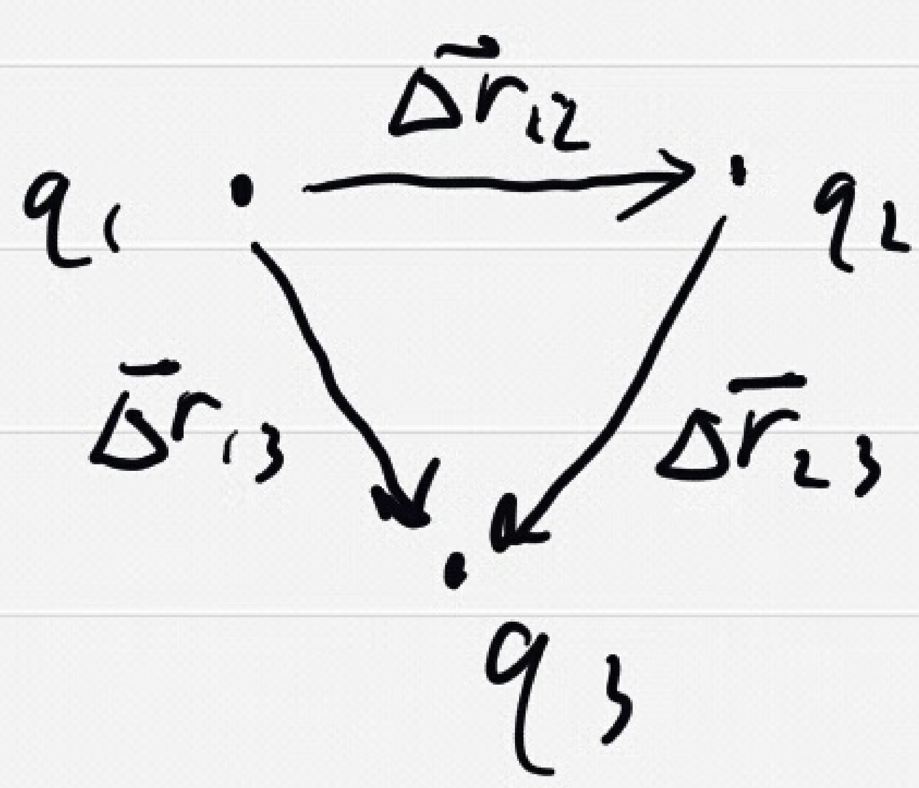
= work to move charge from infinity

Energy of Charge Distribution

$\cdot q_1$ $\quad W_1 = 0$ since $V = 0$



$$W_2 = q_2 \cdot \frac{q_1}{4\pi\varepsilon_0 \Delta r_{12}}$$



$$W_3 = q_3 \cdot \left(\frac{q_1}{4\pi\varepsilon_0 \Delta r_{13}} + \frac{q_2}{4\pi\varepsilon_0 \Delta r_{23}} \right)$$

$$W = W_1 + W_2 + W_3$$
$$= \frac{1}{4\pi\varepsilon_0} \left[\frac{q_1 q_2}{\Delta r_{12}} + \frac{q_1 q_3}{\Delta r_{13}} + \frac{q_2 q_3}{\Delta r_{23}} \right]$$

General: $W_{total} = \frac{1}{4\pi\varepsilon_0} \sum_i \sum_{j>i} \frac{q_i q_j}{\Delta r_{ij}}$

$$= \frac{1}{8\pi\varepsilon_0} \sum_i \sum_{j\neq i} \frac{q_i q_j}{\Delta r_{ij}}$$

$$= \frac{1}{2} \sum_i q_i \sum_{j\neq i} q_j / (4\pi\varepsilon_0 \Delta r_{ij})$$

$$= \frac{1}{2} \sum_i q_i V(\vec{r}_i)$$

w/ $V(\vec{r}_i)$ due to all $q_{j\neq i}$