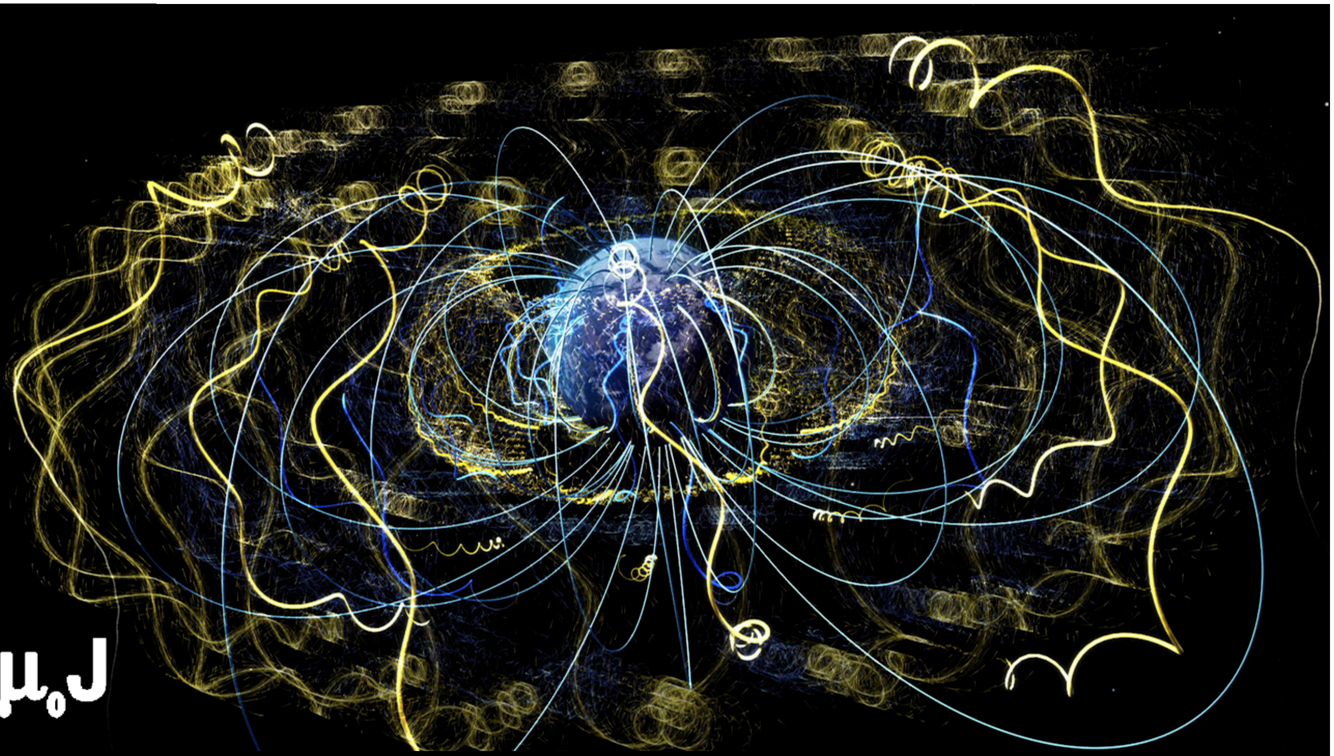
$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}$$



Electricity and Magnetism I: 3811

Professor Jasper Halekas
Van Allen 301
MWF 9:30-10:20 Lecture

Image Charges

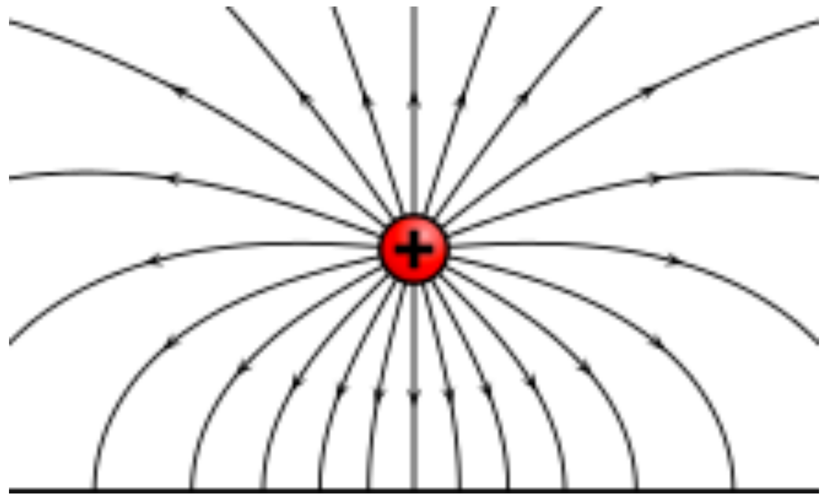


Image Charges

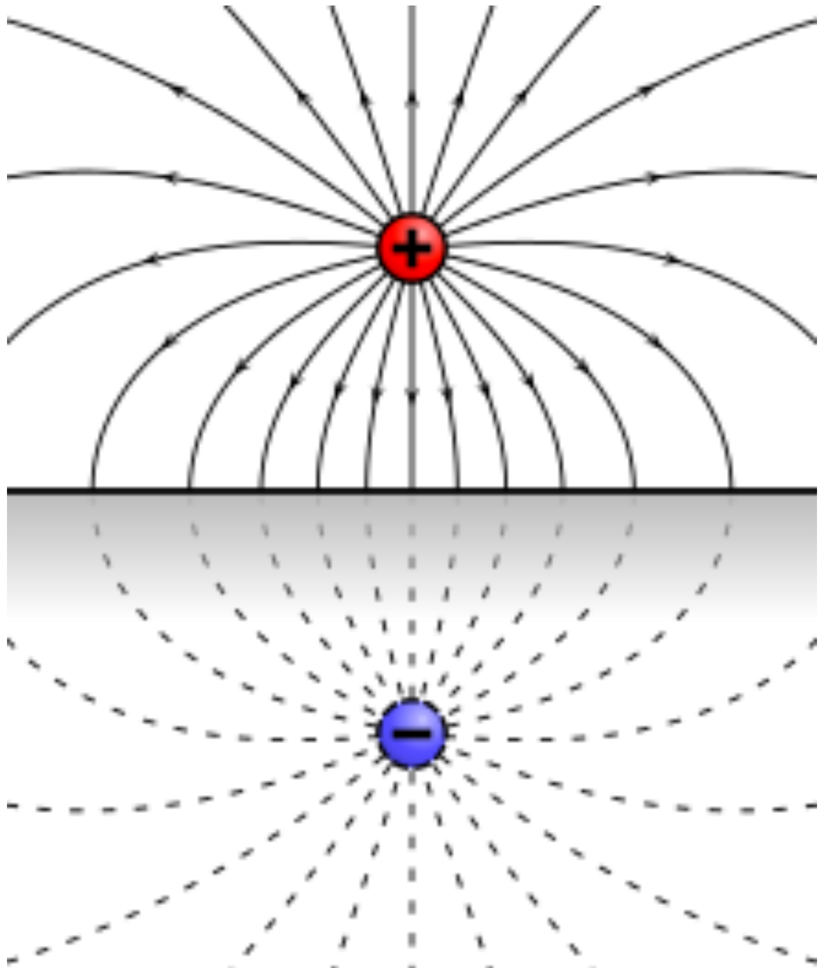
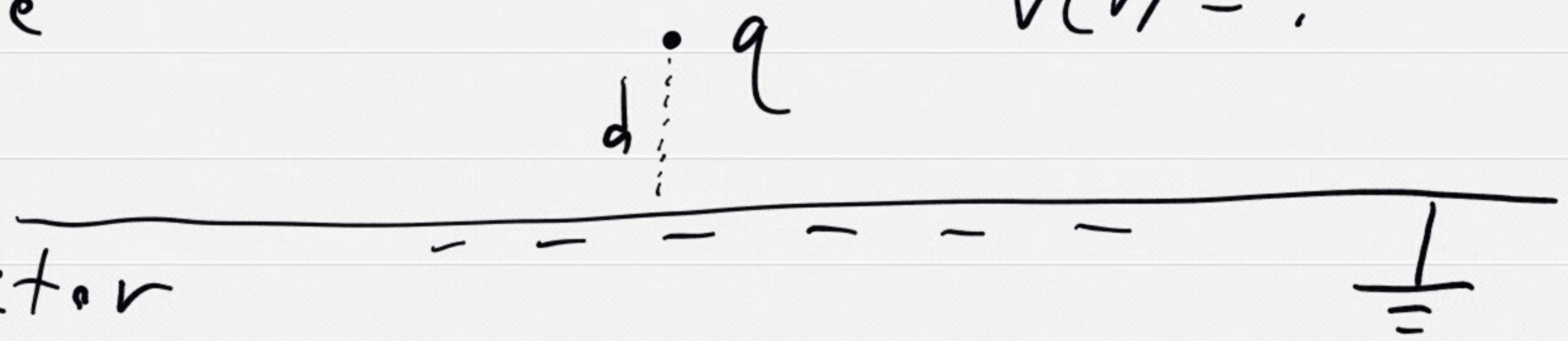


Image Charges

solve $V(\vec{r}) = ?$



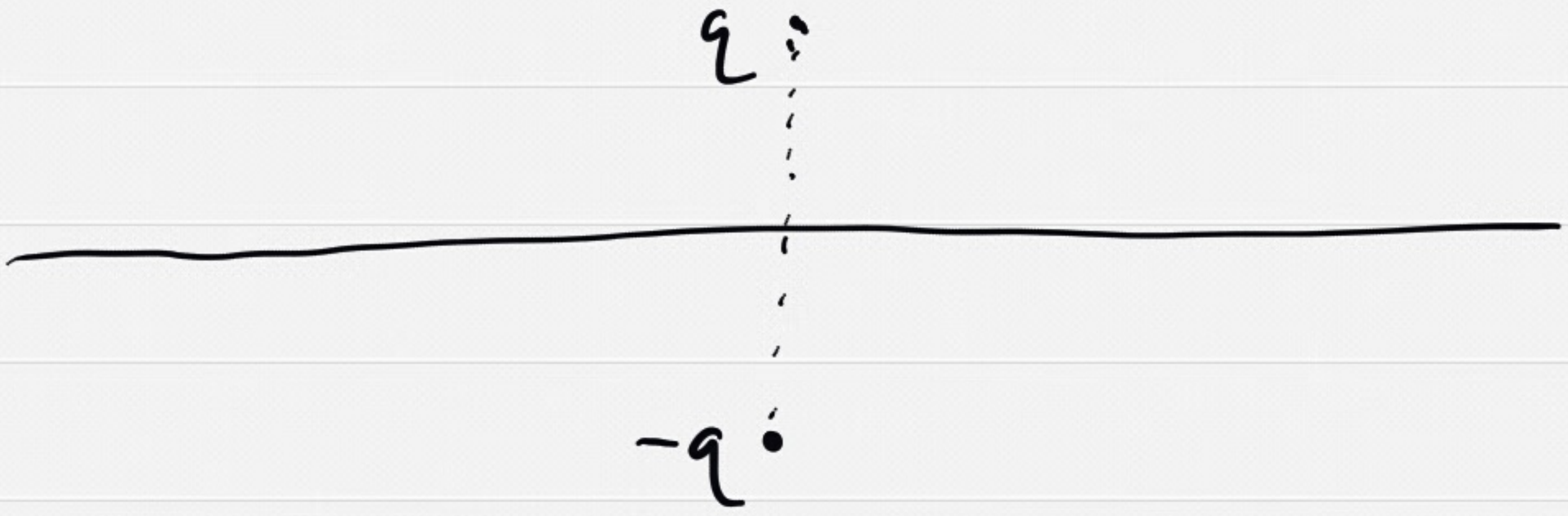
conductor

Constraints: $V = 0$ in conductor
$V = 0$ @ ∞

First uniqueness theorem
→ only one solution

If you can guess it, you win!

Consider instead:



$$V = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\Delta r_+} - q/\Delta r_-\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

- satisfies all boundary conditions!
- It's the answer!
- Valid for $z > 0$ only

Induced Surface Charge

$$\Delta \vec{E} = \sigma/\varepsilon_0 \, \hat{n} = -\partial V/\partial n \, \hat{n}$$

$$\Rightarrow \sigma = -\varepsilon_0 \, \partial V/\partial n$$

$$= -\varepsilon_0 \, \partial V/\partial z \Big|_{z=0}$$

$$-\varepsilon_0 \, \partial V/\partial z = -\frac{1}{4\pi}\left[\frac{-\frac{1}{2}q \cdot 2(z-d)}{(x^2+y^2+(z-d)^2)^{3/2}} - \frac{-\frac{1}{2}q \cdot 2(z+d)}{(x^2+y^2+(z+d)^2)^{3/2}}\right]\Bigg|_0$$

$$= -\frac{1}{4\pi}\left[\frac{qd}{(x^2+y^2+d^2)^{3/2}} + \frac{qd}{(x^2+y^2+d^2)^{3/2}}\right]$$

$$= \frac{-qd}{2\pi} \cdot \frac{1}{(x^2+y^2+d^2)^{3/2}}$$

$$Q = \int \sigma \, da = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sigma(x,y) \, dx \, dy$$

$$= \int_0^{\infty}\int_0^{2\pi} \sigma(r,\varphi) \cdot r \, dr \, d\varphi$$

$$= \int_0^{\infty}\int_0^{2\pi} \cdot \frac{-qd}{2\pi} \cdot \frac{1}{(r^2+d^2)^{3/2}} \cdot r \, dr \, d\varphi$$

$$= -qd \int_0^{\infty} \frac{r}{(r^2+d^2)^{3/2}} \, dr$$

$$= -qd \cdot \frac{-1}{\sqrt{r^2+d^2}}\Bigg|_0^{\infty} = \boxed{-q}$$

- Total induced charge cancels imposed charge

$$\vec{F} = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{(2d)^2}\hat{z}$$

(could get from $\vec{F} = q\vec{E}$
$= -q\nabla V$, but know it
should be same as image
charge analog)

$$W_{on} = \int_{\infty}^{d} \vec{F}_{on}\cdot d\vec{\ell} = \frac{1}{4\pi\varepsilon_0}\int_{\infty}^{d}\frac{q^2}{(2z)^2}dz$$

$$= \frac{1}{4\pi\varepsilon_0}\left.-\frac{q^2}{4z}\right|_{\infty}^{d} = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{4d}$$

Note $W_{on} = \frac{1}{2}\left[-\frac{1}{4\pi\varepsilon_0}\frac{q^2}{\Delta r}\right]$

— we only had to do half
the work of the case w/
two point charges (because
one is imaginary!)

Image Charges

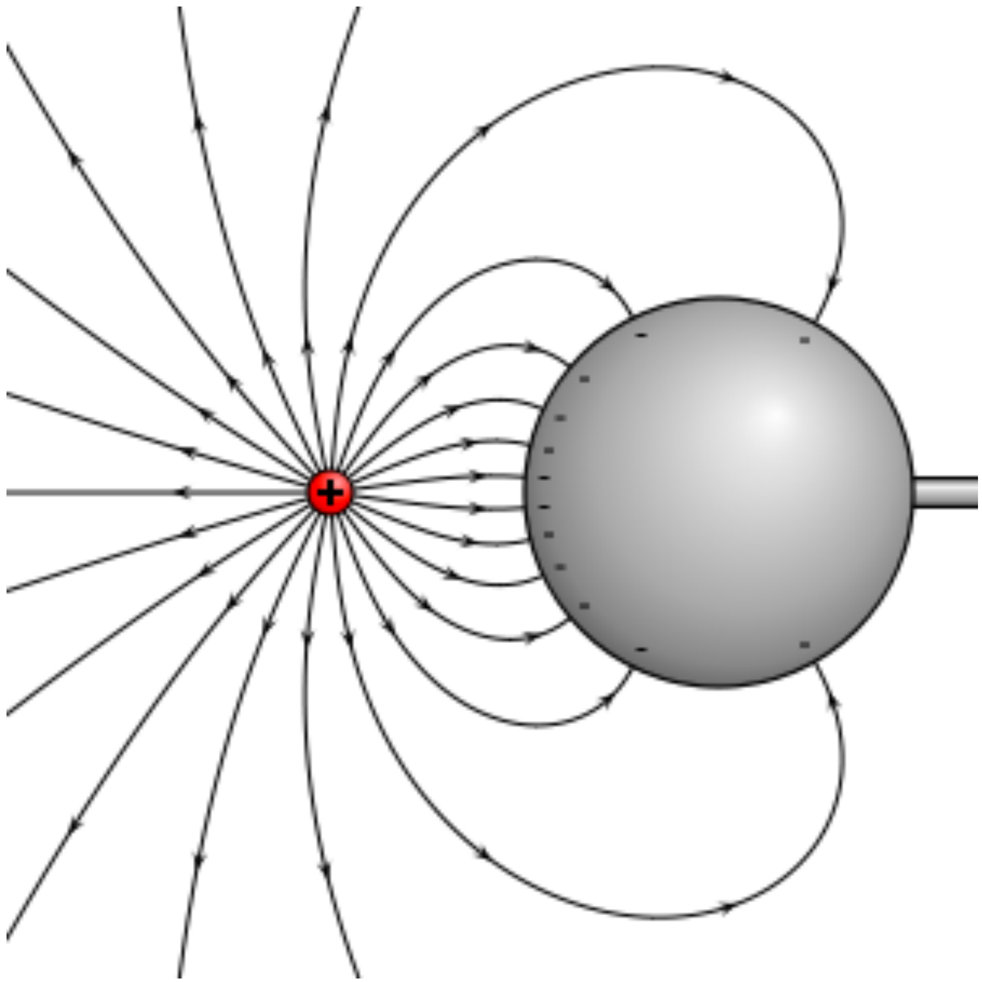
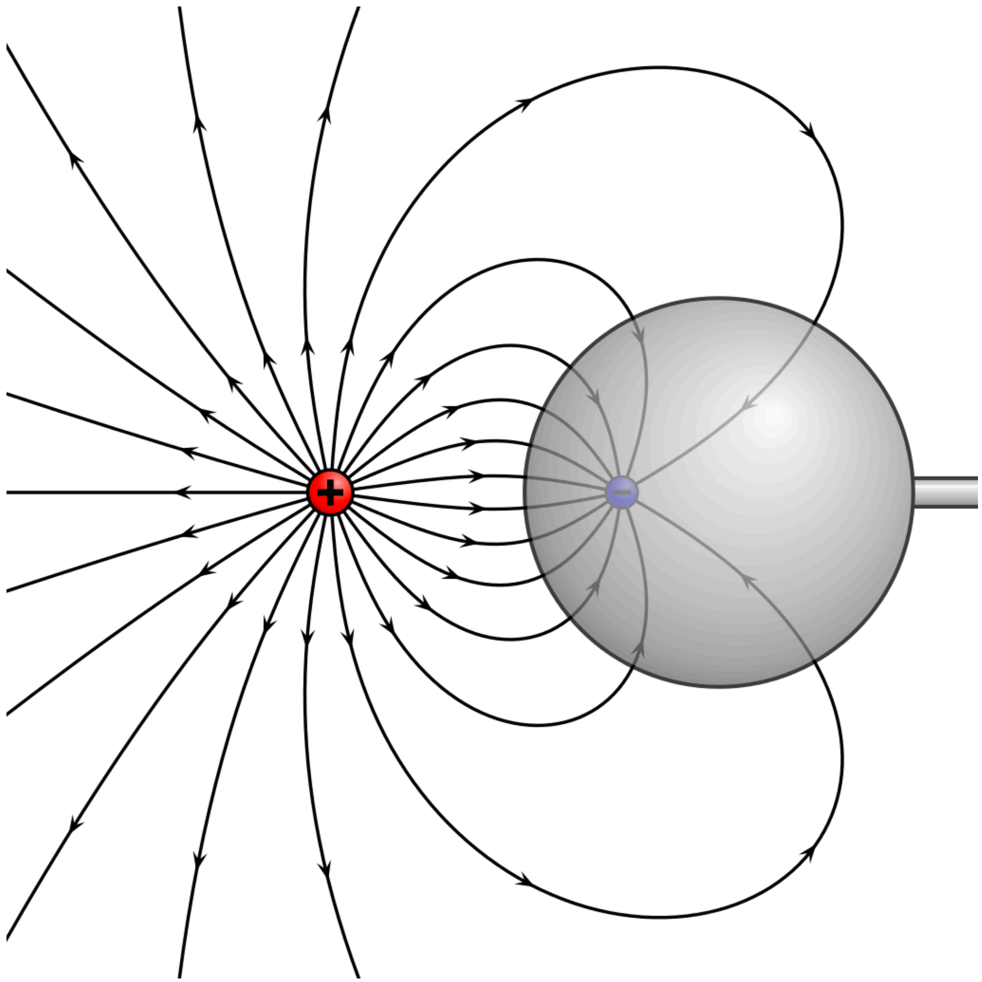
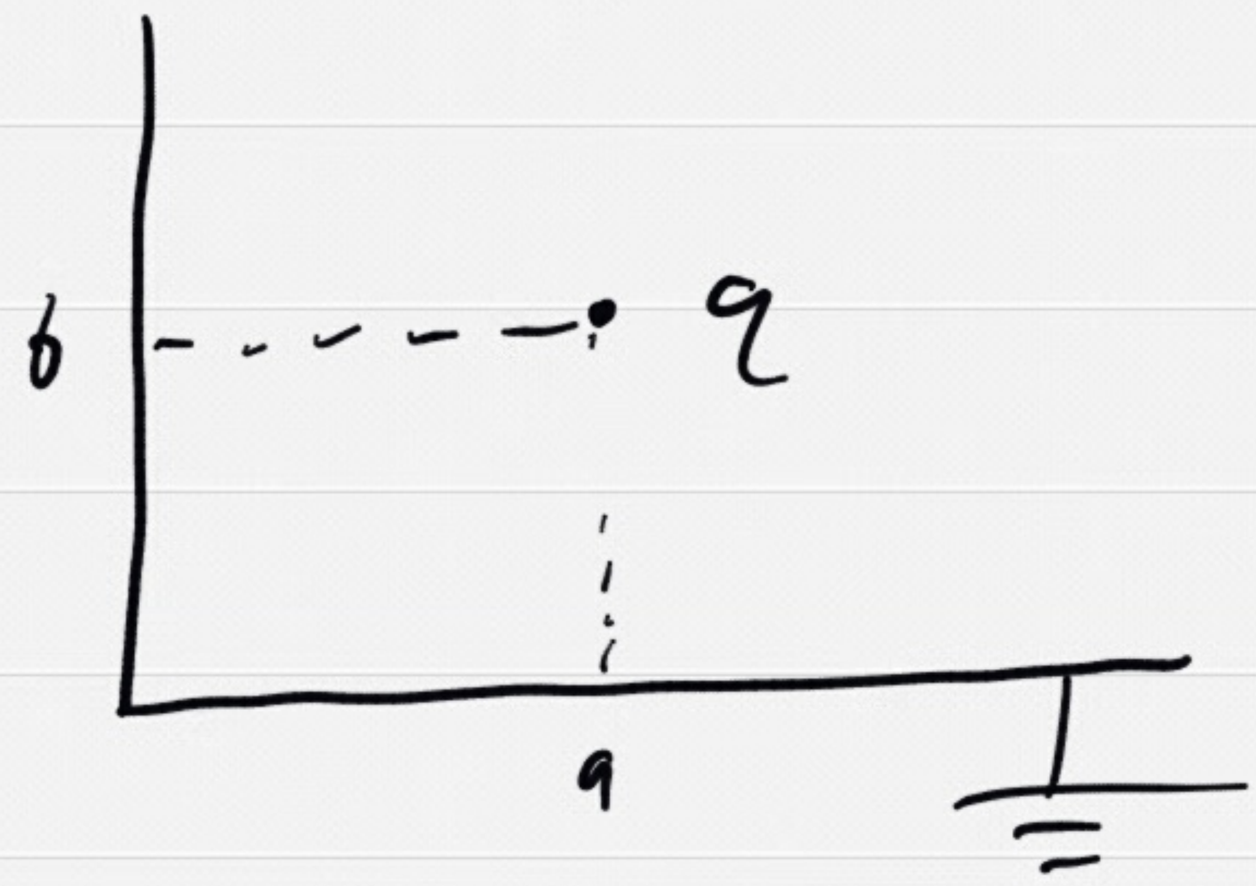


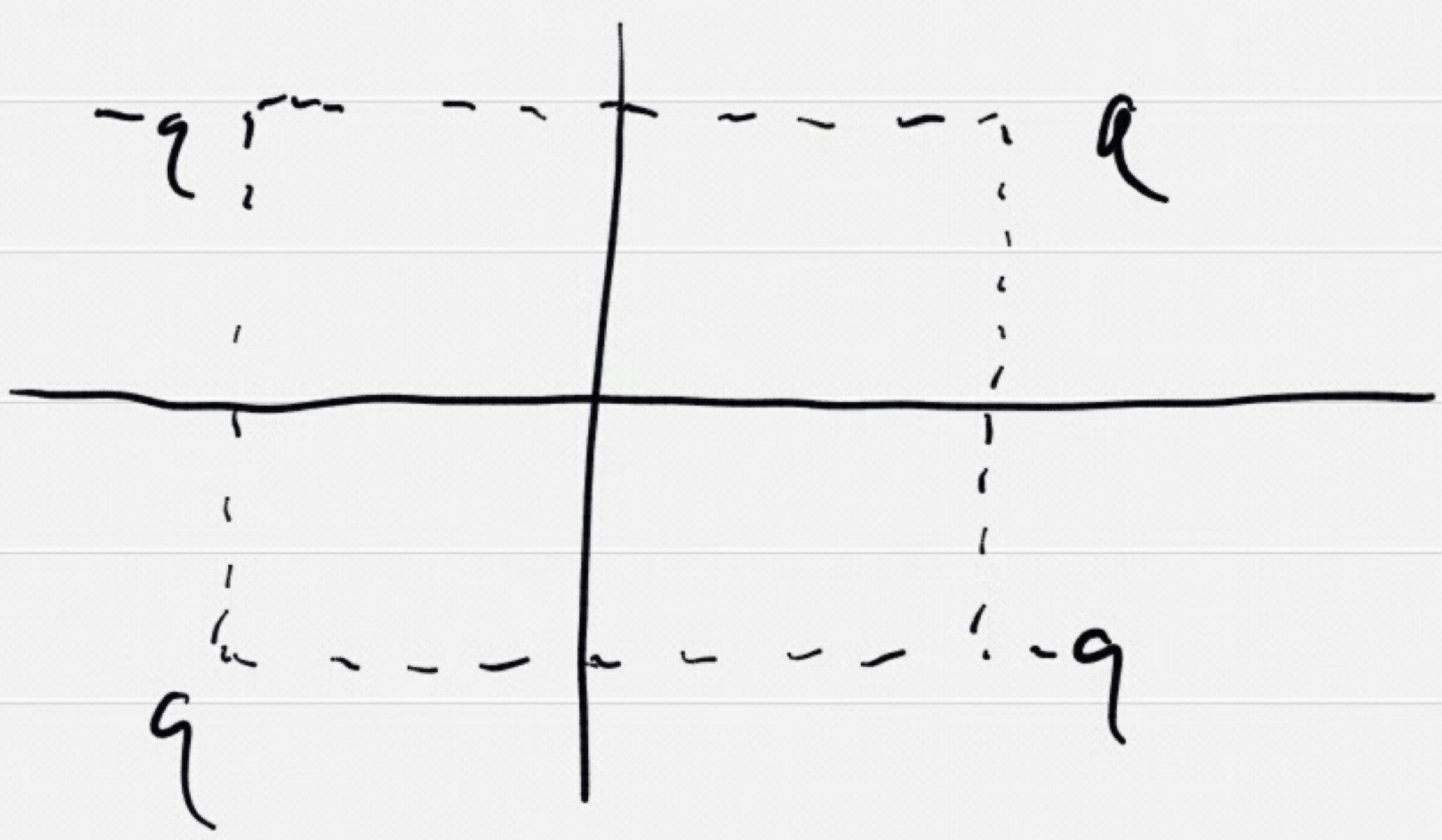
Image Charges



Another case



Equivalent to



$$V(x,y,z) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{(x-a)^2+(y-b)^2+z^2}} + \frac{1}{\sqrt{(x+a)^2+(y+b)^2+z^2}} - \frac{1}{\sqrt{(x-a)^2+(y+b)^2+z^2}} - \frac{1}{\sqrt{(x+a)^2+(y-b)^2+z^2}}\right]$$

$$V(x=0) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{a^2+(y-b)^2+z^2}} + \frac{1}{\sqrt{a^2+(y+b)^2+z^2}} - \frac{1}{\sqrt{a^2+(y+b)^2+z^2}} - \frac{1}{\sqrt{a^2+(y-b)^2+z^2}}\right]$$

$= 0$ & similarly for $V(y=0)$

$$\vec{F} = \frac{q^2}{4\pi\varepsilon_0}\left[-\frac{1}{(2a)^2}\hat{x} - \frac{1}{(2b)^2}\hat{y} + \frac{1}{\left(2\sqrt{a^2+b^2}\right)^2}\left(\cos\theta\,\hat{x} + \sin\theta\,\hat{y}\right)\right]$$

$$\cos\theta = a/\sqrt{a^2+b^2}, \quad \sin\theta = b/\sqrt{a^2+b^2}$$

$$\Rightarrow \vec{F} = \frac{q^2}{16\pi\varepsilon_0}\left[\left(\frac{a}{(a^2+b^2)^{3/2}} - \frac{1}{a^2}\right)\hat{x} + \left(\frac{b}{(a^2+b^2)^{3/2}} - 1/b^2\right)\hat{y}\right]$$