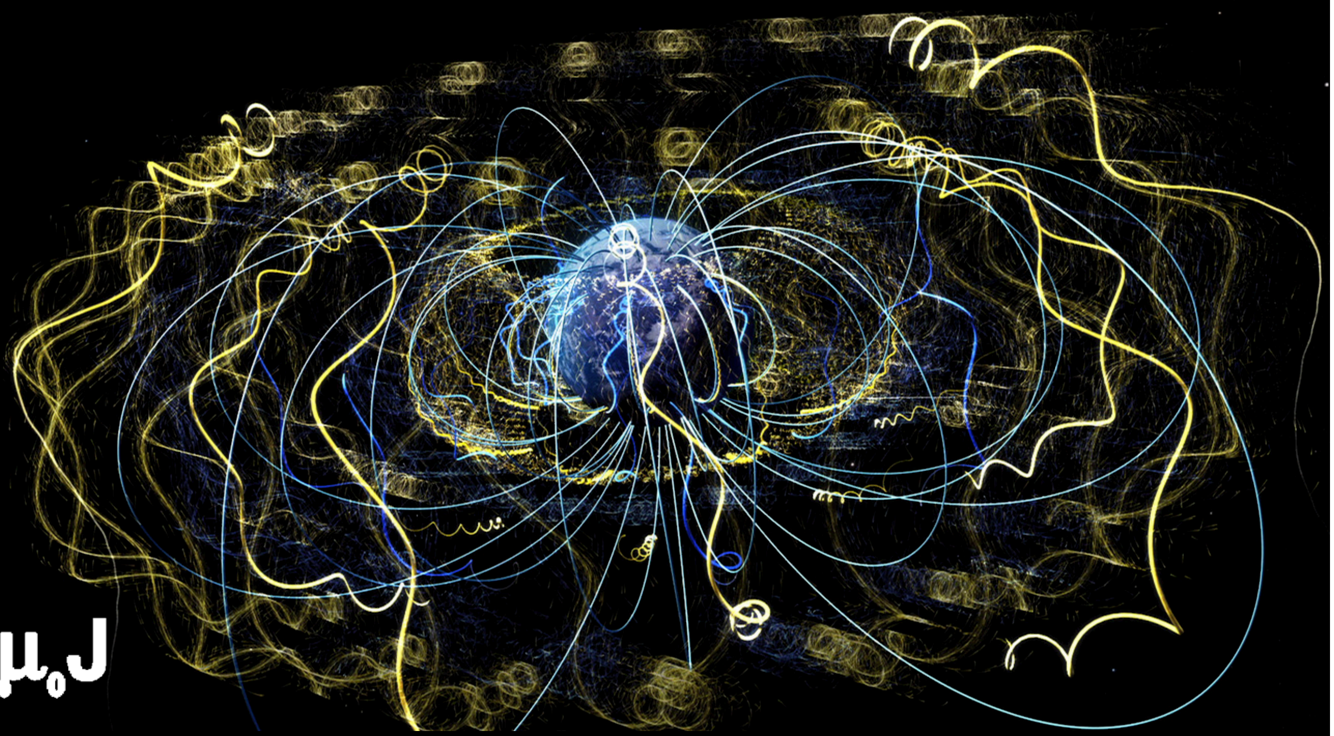
$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

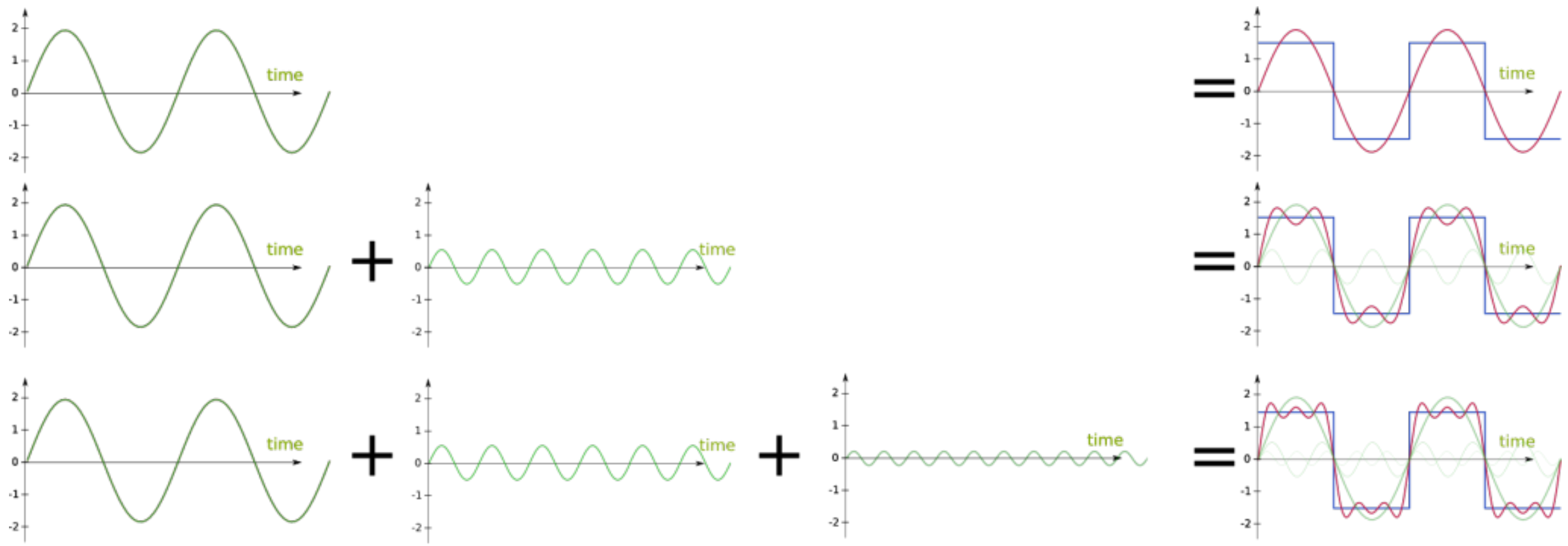
$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}$$



Electricity and Magnetism I: 3811

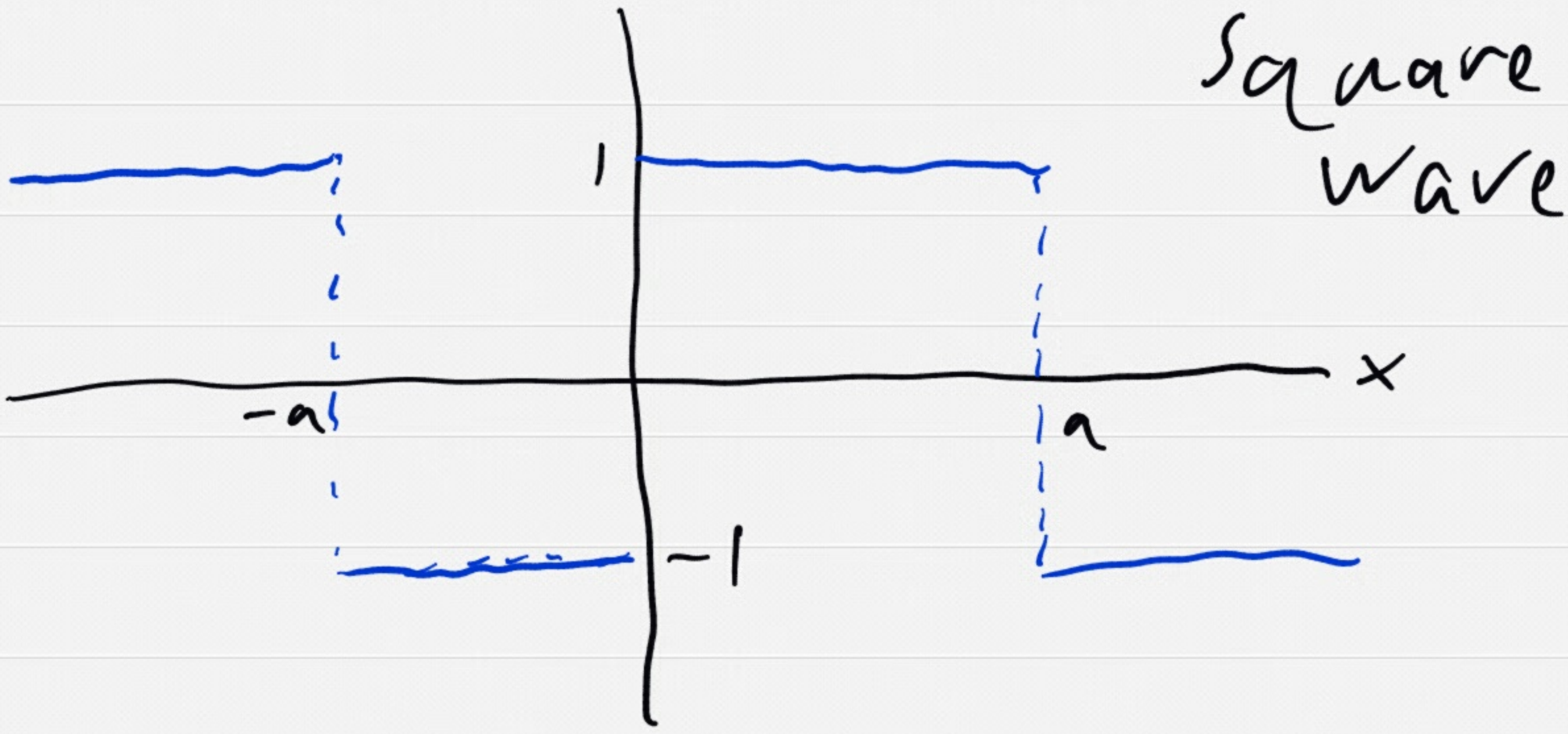
Professor Jasper Halekas
Van Allen 301
MWF 9:30-10:20 Lecture

Fourier Analysis



And so on...

Fourier Analysis



① Pick Basis Functions

$\sin(n\pi x/a)$, $\cos\left(\frac{n\pi x}{a}\right)$

periodic w/ period $2a$

- square wave is odd so only sin terms contribute

② Express function as sum

$$f(x) = \sum_{1}^{\infty} a_n \sin\left(\frac{n\pi x}{a}\right)$$

③ Compute integrals

$$\int_{-a}^{a} f(x) \sin\left(\frac{m\pi x}{a}\right) dx$$

$$= \sum_{1}^{\infty} a_n \int_{-a}^{a} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi x}{a}\right) dx$$

$$\int_{-a}^{a} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi x}{a}\right) dx$$

$$= \int_{-a}^{a} \frac{1}{2}\left[\cos\left(\frac{(n-m)\pi x}{a}\right) - \cos\left(\frac{(n+m)\pi x}{a}\right)\right] dx$$

$= 0$ unless $n = m$ since cos is even

$= \int_{-a}^{a} \frac{1}{2}\, dx = a$ if $n = m$

$$\Rightarrow \int_{-a}^{a} f(x) \sin\left(\frac{m\pi x}{a}\right) dx$$

$$= a \cdot a_m$$

(4) $$a_m = \frac{1}{a} \int_{-a}^{a} f(x) \cdot \sin\left(\frac{m\pi x}{a}\right) dx$$

"Fourier's Trick"

- works because basis functions are orthogonal over interval $-a$ to a

$$\int_{-a}^{a} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi x}{a}\right) dx$$

$= 0 \quad n \neq m$

Square Wave Coefficients

$$f(x) = \begin{cases} -1 & -a < x < 0 \\ +1 & 0 < x < a \end{cases}$$

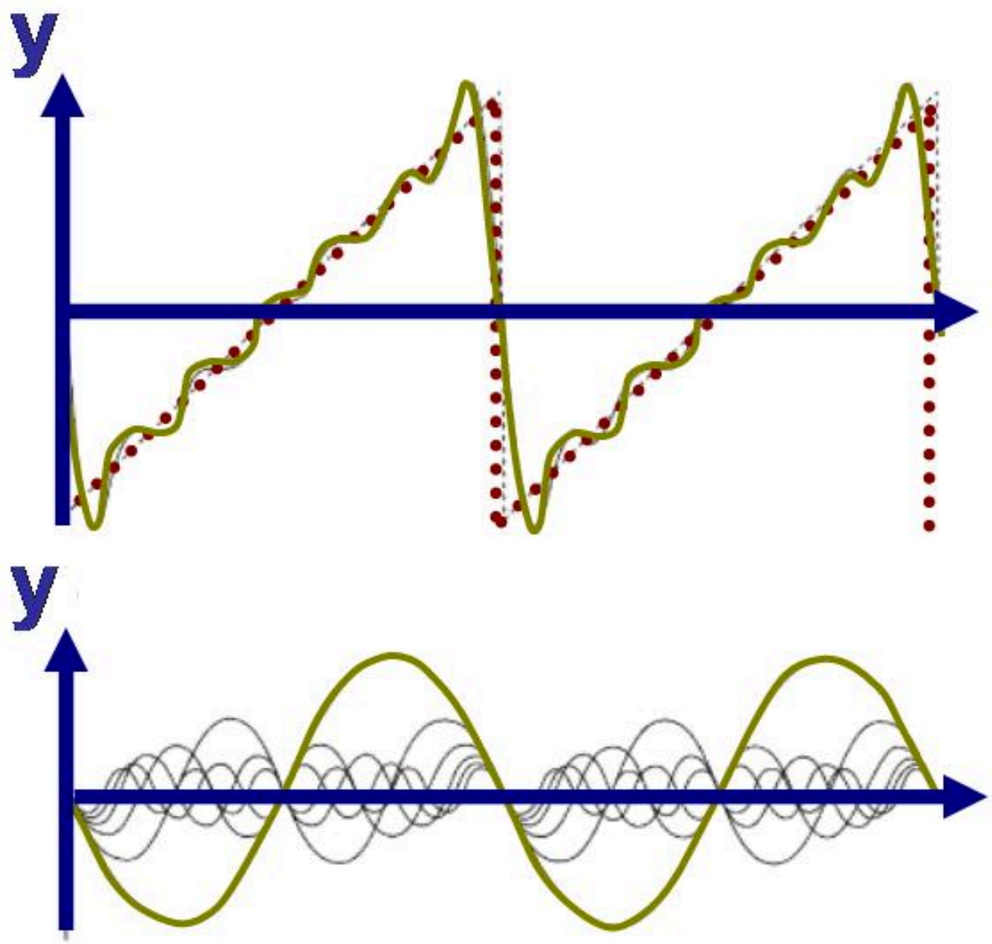
$$\Rightarrow a_m = \frac{1}{a}\left[\int_{-a}^{0} -\sin\left(\frac{m\pi x}{a}\right)dx + \int_{0}^{a} \sin\left(\frac{m\pi x}{a}\right)dx\right]$$

$$= \frac{1}{a}\left[\frac{a}{m\pi}\cos\left(\frac{m\pi x}{a}\right)\Big|_{-a}^{0} - \frac{a}{m\pi}\cos\left(\frac{m\pi x}{a}\right)\Big|_{0}^{a}\right]$$

$$= \begin{cases} 4/m\pi & \text{for } m \text{ odd} \\ 0 & \text{for } m \text{ even} \end{cases}$$

$$\Rightarrow \boxed{f(x) = 4/\pi \sum_{n\,\text{odd}} \frac{\sin(n\pi x)}{n}}$$

Fourier Analysis



Orthogonality

- *Orthogonal basis*

$$\int_{-\pi}^{\pi} \cos mx \sin nx \, dx = 0$$

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \neq 0 \\ 2\pi & \text{if } m = n = 0 \end{cases}$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \neq 0 \\ 0 & \text{if } m = n = 0 \end{cases}$$

Completeness

- $f(x)$ defined for a range of values of x of length 2π, say $-\pi \leq x \leq \pi$;

  we approximate this function as an infinite sum of trigonometric functions, as **Fourier series** for $f(x)$,

$$f(x) \approx S(x) \equiv \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

- where a_n, b_n are an infinite series of constants to be determined called the **Fourier coefficients**.
- the $\frac{1}{2}a_0$ is really a $\cos 0x = 1$ constant term, and the $\frac{1}{2}$ is put in for convenience

Completeness: **Any** function can be expanded in this way!

Separation of Variables

$$\nabla^2 V = 0 \quad \text{if } \rho = 0$$

$$\partial^2 V/\partial x^2 + \partial^2 V/\partial y^2 + \partial^2 V/\partial z^2 = 0$$

- Look for solution of form

$$V(x,y,z) = X(x)\,Y(y)\,Z(z)$$

$$\Rightarrow \partial^2 X/\partial x^2\, YZ + X\, \partial^2 Y/\partial y^2\, Z + XY\, \partial^2 Z/\partial z^2 = 0$$

$$\Rightarrow \frac{1}{X}\partial^2 X/\partial x^2 + \frac{1}{Y}\partial^2 Y/\partial y^2 + \frac{1}{Z}\partial^2 Z/\partial z^2 = 0$$

Must hold for all x, y, z

$$\Rightarrow \frac{1}{X}\partial^2 X/\partial x^2 = C_1$$

$$\frac{1}{Y}\partial^2 Y/\partial y^2 = C_2$$

$$\frac{1}{Z}\partial^2 Z/\partial z^2 = C_3$$

w/ $C_1 + C_2 + C_3 = 0$

If $C_1 < 0$:

$$\frac{1}{X}\,\partial^2 X / \partial x^2 = -k^2$$

$$\partial^2 X / \partial x^2 = -k^2 X$$

$$\Rightarrow X(x) = e^{ikx},\ e^{-ikx},$$

or combinations like $\cos(kx)$, $\sin(kx)$

If $C_1 > 0$:

$$\partial^2 X / \partial x^2 = k^2 X$$

$$\Rightarrow X(x) = e^{kx},\ e^{-kx},$$

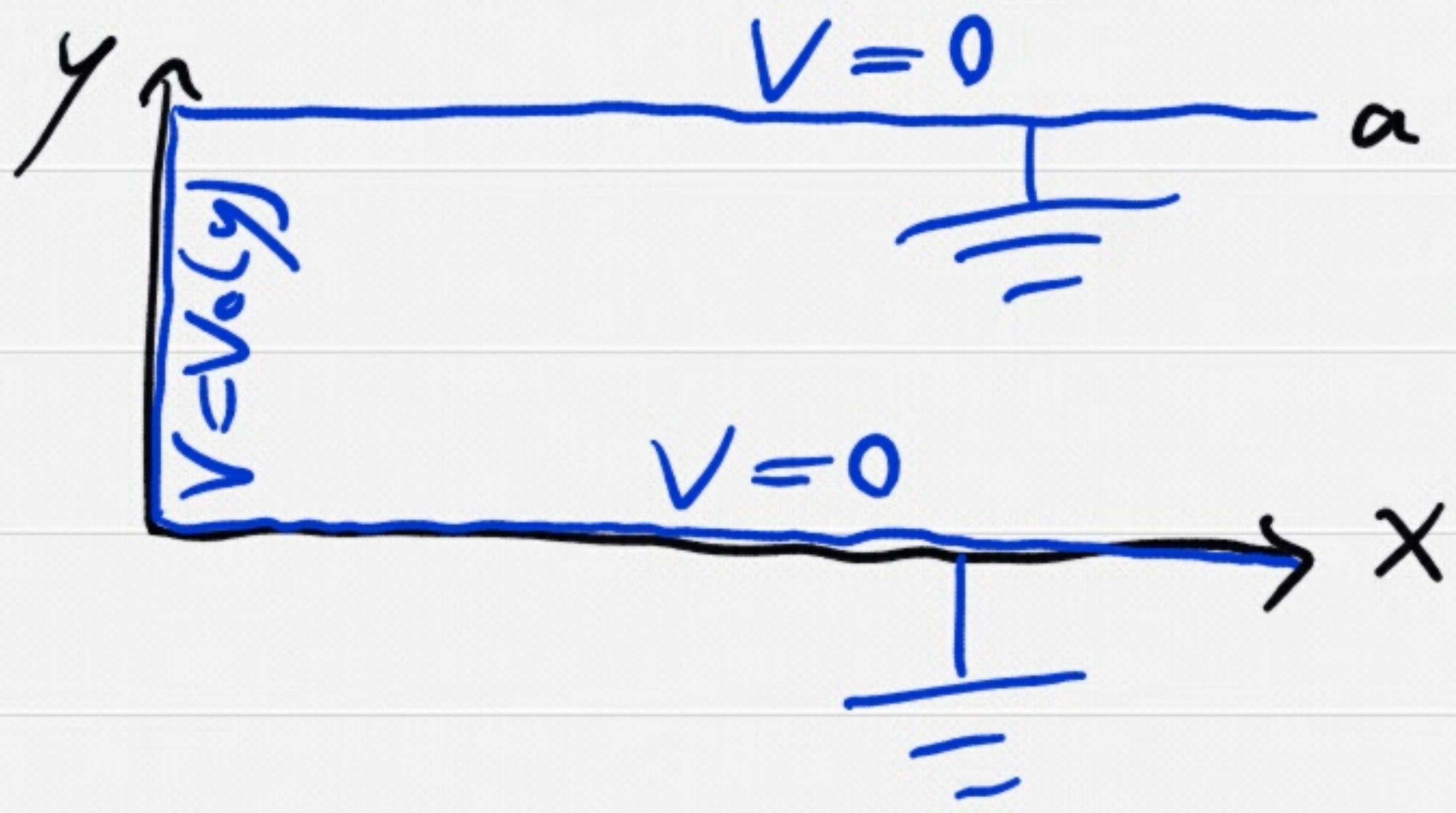
or combinations like $\cosh(kx)$, $\sinh(kx)$

- similarly for Y, Z

$$C_1 + C_2 + C_3 = 0$$

$\Rightarrow$ Not all positive or negative

2-d Example:



$V = 0$ @ $y = 0$, $y = a$

$V = V_0(y)$ @ $x = 0$

$V \to 0$ @ $x = \infty$

Solutions:

$X(x) = Ae^{-\kappa x}$ is only valid solution

$\Rightarrow C_1 > 0$

$\Rightarrow C_2 < 0$

$Y(y) = B\sin\left(\frac{n\pi y}{a}\right)$ matches boundary conditions

So $C_2 = -\frac{n^2\pi^2}{a^2} \Rightarrow C_1 = +\frac{n^2\pi^2}{a^2}$

$\Rightarrow \kappa = \frac{n\pi}{a}$

Solutions: $\boxed{V_n(x,y) = C_n e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)}$

Salution fon particular

$V_0(y)$:

$$V(x,y) = \sum_n C_n e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)$$

$$V(0,y) = V_0(y)$$
$$= \sum_n C_n \sin\left(\frac{n\pi y}{a}\right)$$

$$\int_0^a V_0(y) \sin\left(\frac{m\pi y}{a}\right) dy = \sum_n C_n \int_0^a \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{m\pi y}{a}\right) dy$$

$$= a/2 \cdot C_m$$

$$\Rightarrow C_m = \frac{2}{a} \int_0^a V_0(y) \sin\left(\frac{m\pi y}{a}\right) dy$$