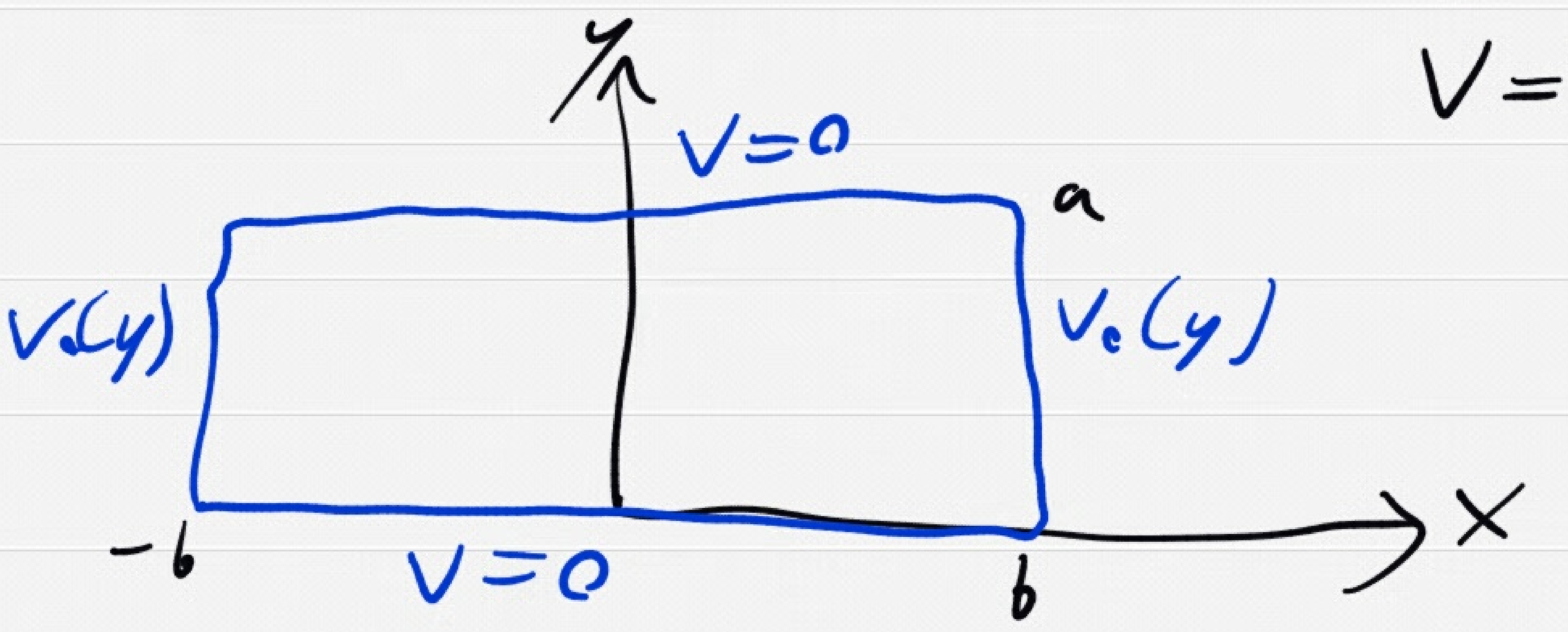
2-d Example #2

$V=0$ @ $y=0,a$
$V=V_0(y)$ @ $x=\pm b$



Solutions:

$$Y(y) = A\sin\left(\frac{n\pi y}{a}\right)$$

$$\partial^2 Y/\partial y^2 = C_2 Y = -k_y^2 Y$$

$$\Rightarrow k_y = \frac{n\pi}{a}$$

$$X(x) = Be^{k_x x} + Ce^{-k_x x}$$

$$\partial^2 X/\partial x^2 = C_1 X = k_x^2 X$$

$$C_1 = -C_2 = k_y^2 = k_x^2 = k^2$$

Boundary conditions symmetric

$$\Rightarrow B = C$$

$$\Rightarrow X(x) = B(e^{kx} + e^{-kx})$$

$$= 2B\cosh(kx)$$

$$\Rightarrow \boxed{V_n(x,y) = C_n\cosh\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)}$$

$$V(x,y) = \sum_n C_n\cosh\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)$$

$$V(x=\pm b, y) = V_0(y)$$

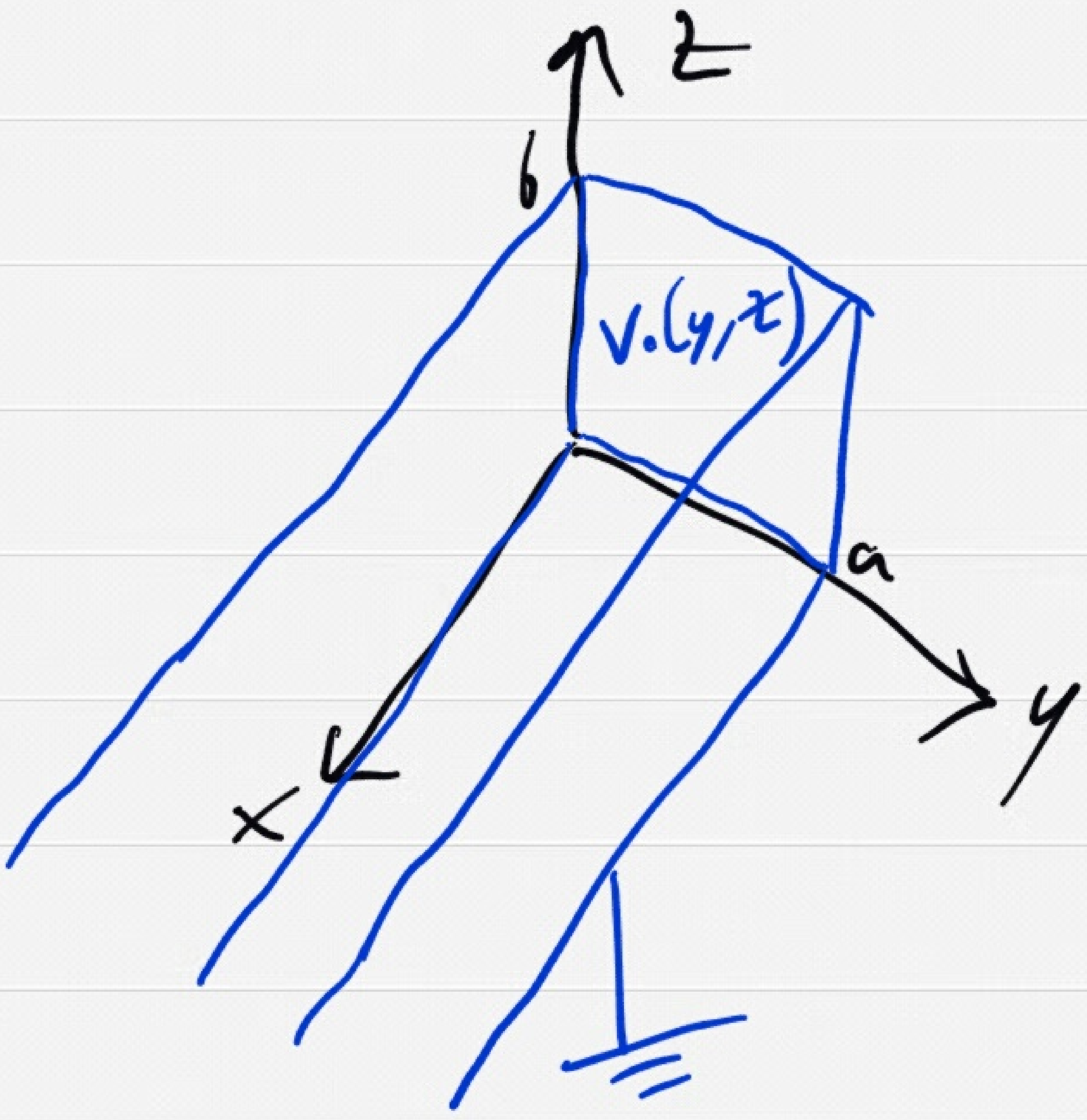
$$= \sum_n C_n \cosh\left(\frac{n\pi b}{a}\right) \sin\left(\frac{n\pi y}{a}\right)$$

$$\int_0^a V_0(y) \sin\left(\frac{m\pi y}{a}\right) dy$$

$$= C_m \cdot \frac{a}{2} \cdot \cosh\left(\frac{m\pi b}{a}\right)$$

$$\Rightarrow C_m = \frac{2}{a \cosh\left(\frac{m\pi b}{a}\right)} \int_0^a V_0(y) \sin\left(\frac{m\pi y}{a}\right) dy$$

3-d Example



$V = 0$ @ $y = 0, a$

$V = 0$ @ $z = 0, b$

$V = V_0(y,z)$ @ $x = 0$

$V \to 0$ @ $x = \infty$

Solutions:

$$X(x) = A e^{-k_x x}$$

$$Y(y) = B \sin\left(\frac{n_y \pi y}{a}\right)$$

$$Z(z) = C \sin\left(\frac{n_z \pi z}{b}\right)$$

$$C_1 + C_2 + C_3 = 0$$

$$\Rightarrow \quad k_x^2 - \frac{n_y^2 \pi^2}{a^2} - \frac{n_z^2 \pi^2}{b^2} = 0$$

$$\Rightarrow \quad k_x = \pi \sqrt{n_y^2/a^2 + n_z^2/b^2}$$

$$\boxed{V_{n_y n_z}(x,y,z) = C_{n_y n_z} e^{-k_x x} \sin\left(\frac{n_y \pi y}{a}\right) \sin\left(\frac{n_z \pi z}{b}\right)}$$

$$V(x,y) = \sum_{n_y} \sum_{n_z} V_{n_y n_z}(x,y,z)$$

$$V(a,y,z) = \sum_{n_y}\sum_{n_z} C_{n_y n_z} \sin\left(\frac{n_y \pi y}{a}\right) \sin\left(\frac{n_z \pi z}{b}\right)$$

$$= V_0(y,z)$$

$$\int_0^a \int_0^b V_0(y,z) \sin\left(\frac{m_y \pi y}{a}\right) \sin\left(\frac{m_z \pi z}{b}\right) dy\, dz$$

$$= \frac{a}{2}\,\frac{b}{2}\, C_{m_y m_z}$$

$$\Rightarrow C_{m_y m_z} = \frac{2}{a}\,\frac{2}{b} \int_0^a \int_0^b V_0(y,z) \sin\left(\frac{m_y \pi y}{a}\right) \cdot \sin\left(\frac{m_z \pi z}{b}\right) dy\, dz$$

E.g. $V_0(y,z) = \sin\left(\frac{\pi y}{a}\right) \sin\left(\frac{2\pi z}{b}\right)$

$\Rightarrow C_{m_y m_z} = 0$ except $m_y = 1, m_z = 2$

$$C_{12} = \frac{2}{a} \cdot \frac{2}{b} \cdot \frac{a}{2} \cdot \frac{b}{2} = 1$$

$$\Rightarrow V(x,y,z) = e^{-K_x x} \sin\left(\frac{\pi y}{a}\right) \sin\left(\frac{2\pi z}{b}\right)$$

w/ $K_x = \pi \sqrt{1/a^2 + 2^2/b^2}$

$$\Rightarrow V(x,y,z) = e^{-\pi\sqrt{1/a^2 + 4/b^2}\, x} \sin\left(\frac{\pi y}{a}\right) \sin\left(\frac{2\pi z}{b}\right)$$

Laplace's Equation: Sphericals

$$\nabla^2 V = 0$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0$$

Assume azimuthal symmetry

$$\Rightarrow \frac{\partial^2 V}{\partial\phi^2} = 0$$

$$\Rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right) = 0$$

$$\Rightarrow \frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right) = 0$$

Look for solutions $V(r,\theta) = R(r)\,\Theta(\theta)$

$$\Rightarrow \frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right)\Theta + R \cdot \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial\Theta}{\partial\theta}\right) = 0$$

$$\Rightarrow \frac{1}{R}\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) + \frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial\Theta}{\partial\theta}\right) = 0$$

$$f(r) + g(\theta) = 0$$

$$\Rightarrow f(r),\ g(\theta) = \text{const.}$$

$$\Rightarrow \frac{1}{R}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) = C$$

$$\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) = -C$$