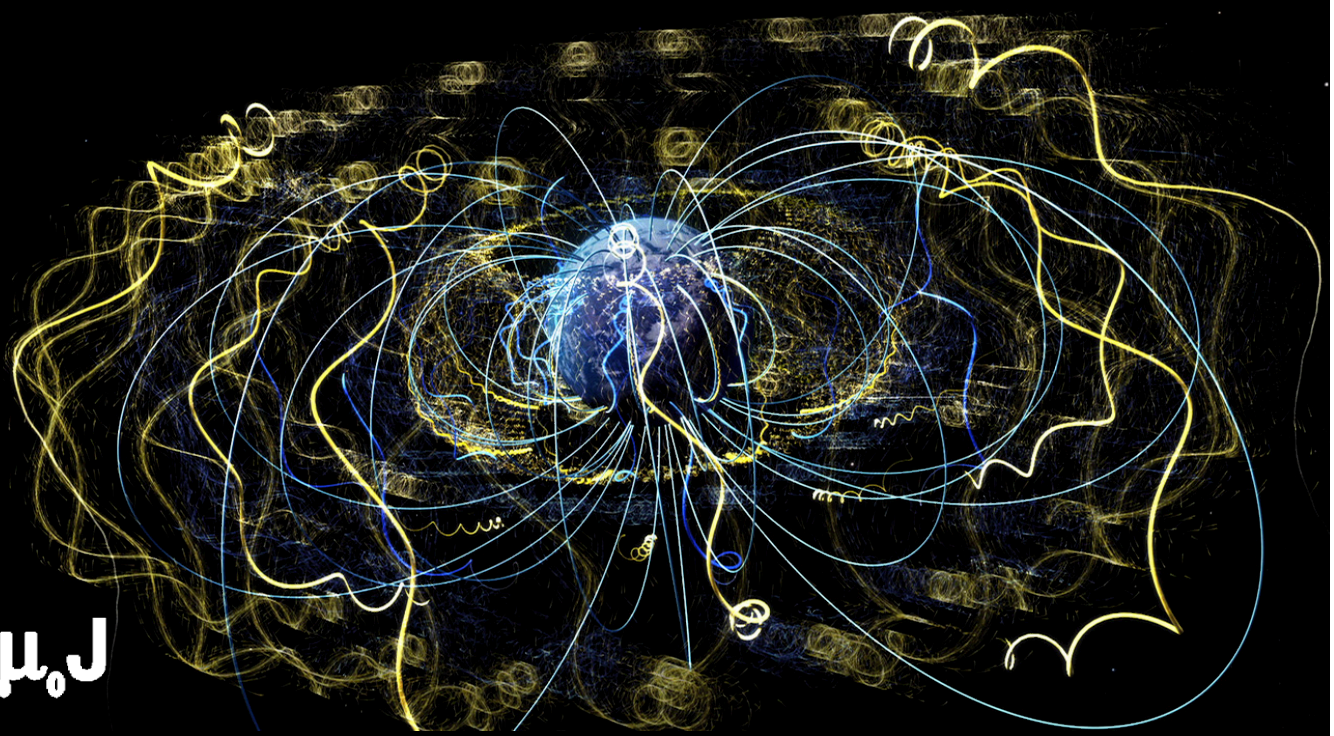
$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}$$



Electricity and Magnetism I: 3811

Professor Jasper Halekas
Van Allen 301
MWF 9:30-10:20 Lecture

convention: $C = \ell(\ell+1)$

$$\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) = \ell(\ell+1) R$$

$$\Rightarrow \boxed{R(r) = A\, r^\ell + \frac{B}{r^{\ell+1}}}$$

Check:

$$\frac{dR}{dr} = A \ell r^{\ell-1} - \frac{B(\ell+1)}{r^{\ell+2}}$$

$$r^2 \frac{dR}{dr} = A \ell r^{\ell+1} - \frac{B(\ell+1)}{r^\ell}$$

$$\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) = A \ell \cdot (\ell+1) r^\ell$$

$$+ \frac{B \cdot \ell \cdot (\ell+1)}{r^{\ell+1}}$$

$$= \ell(\ell+1) \cdot R \quad //$$

$$\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) = -\ell(\ell+1)\sin\theta\, \Theta$$

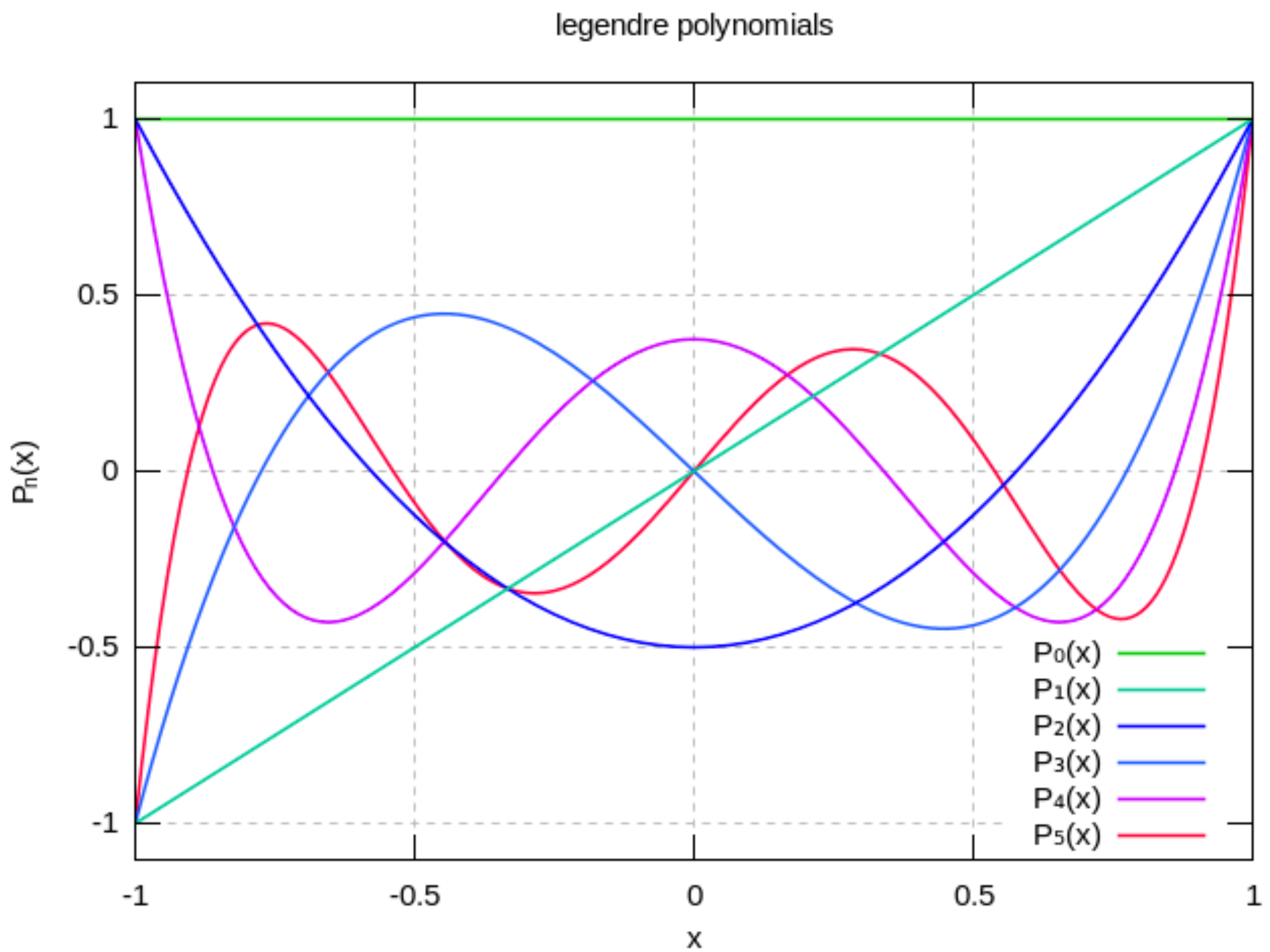
$$\rightarrow \boxed{\Theta(\theta) = P_\ell(\cos\theta)}$$

w/ P_ℓ = "Legendre Polynomial"

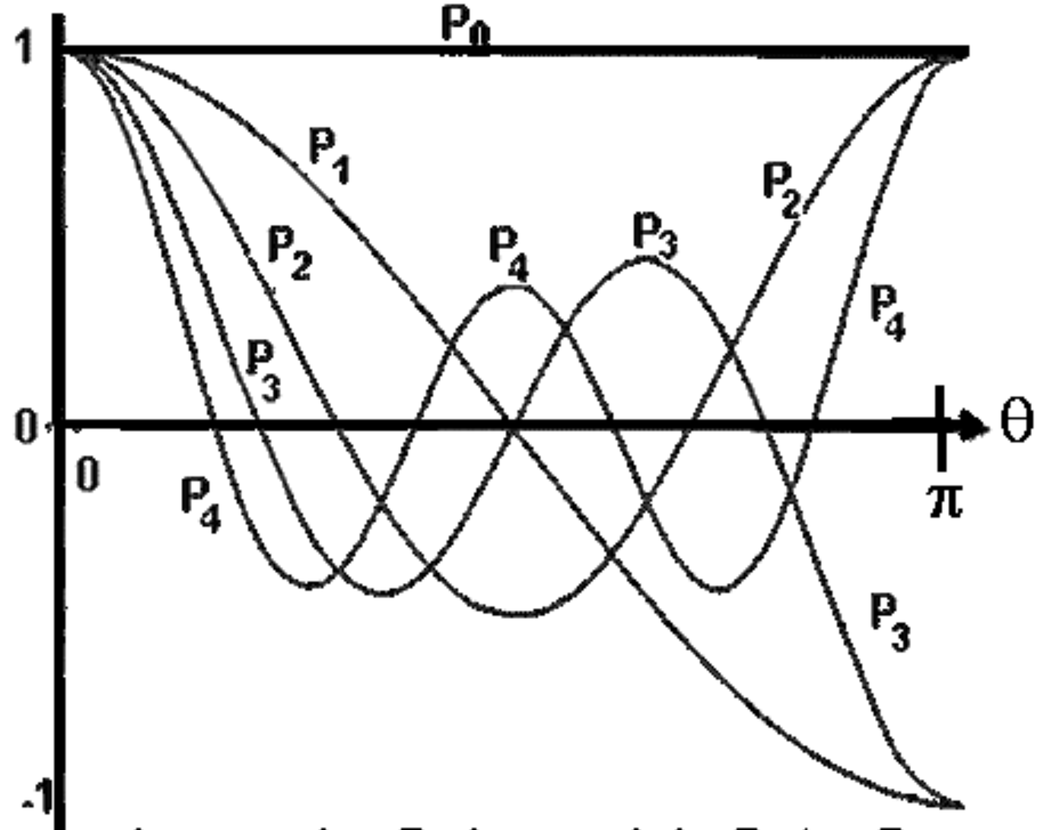
$$V_\ell(r,\theta) = \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)$$

$$V(r,\theta) = \sum_\ell \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)$$

Legendre Polynomials



Legendre Polynomials



Legendre Polynomials P_0 to P_4
Figure A3

Orthagonality

$$\int_{-1}^{1} P_l(x) P_m(x)\, dx$$

$$= \int_0^{\pi} P_l(\cos\theta) P_m(\cos\theta) \cdot \sin\theta\, d\theta$$

$$= \begin{cases} 0 & l \neq m \\ 2/(2l+1) & l = m \end{cases}$$

- Legendre Polynomials are also camplete.

Example: Specified Potential on sphere

Say $V(r=R, \theta) = V_0(\theta)$

$$V(r,\theta) = \sum_l \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$$

$r < R$: $V_<(r,\theta) = \sum_l A_l r^l P_l(\cos\theta)$

$$V_<(R,\theta) = \sum_l A_l R^l P_l(\cos\theta)$$
$$= V_0(\theta)$$

$$\int_0^\pi V_0(\theta) P_m(\cos\theta) \sin\theta d\theta$$

$$= \int_0^\pi \sum_l A_l R^l P_l(\cos\theta) P_m(\cos\theta) \sin\theta d\theta$$

$$= A_m R^m \cdot \frac{2}{2m+1}$$

$$\Rightarrow A_m = \frac{2m+1}{2R^m} \int_0^\pi V_0(\theta) P_m(\cos\theta) \sin\theta d\theta$$

$r > R$: $V_>(r,\theta) = \sum_l \frac{B_l}{r^{l+1}} P_l(\cos\theta)$

$$\int_0^\pi V_0(\theta) P_m(\cos\theta) \sin\theta d\theta$$
$$= \int_0^\pi \sum_l \frac{B_l}{R^{l+1}} P_l(\cos\theta) P_m(\cos\theta) \sin\theta d\theta$$

$$\Rightarrow B_m = \frac{2m+1}{2} R^{m+1} \int_0^\pi V_0(\theta) P_m(\cos\theta) \sin\theta d\theta$$

What if $V_0(\theta) = \text{const.}$
$= V_0$

- Only $m=0$ term contributes. $P_0(\cos\theta) = 1$

So $A_0 = \frac{2\cdot 0+1}{2R^0} \cdot V_0 \cdot \frac{2}{2\cdot 0+1}$

$= V_0$

$\Rightarrow V_<(r,\theta) = V_0$

similarly $B_0 = \frac{2\cdot 0+1}{2} \cdot R^1 \cdot V_0 \cdot \frac{2}{2\cdot 0+1}$
$= V_0 \cdot R$

$\Rightarrow V_>(r,\theta) = \frac{V_0 \cdot R}{r}$

Compare to solution for charged sphere:

$$V = Q/4\pi\varepsilon_0 r \qquad r > R$$
$$= Q/4\pi\varepsilon_0 R \qquad r < R \qquad \checkmark$$

Specified Charge Density

$\sigma_0(\theta)$ on sphere at $r = R$

$$V_<(r,\theta) = \sum_l A_l r^l P_l(\cos\theta)$$

$$V_>(r,\theta) = \sum_l B_l / r^{l+1} \, P_l(\cos\theta)$$

V is continuous, so

$$A_l R^l = B_l / R^{l+1}$$

$$\Rightarrow B_l = A_l R^{2l+1}$$

$$\Delta E = \sigma/\varepsilon_0 = -\partial V_> / \partial r - \left(-\partial V_< / \partial r \right) \Big|_{r=R}$$

$$= \sum_l l A_l R^{l-1} P_l(\cos\theta)$$

$$+ \sum_l (l+1) B_l / R^{l+2} \, P_l(\cos\theta)$$

$$= \sum_l (2l+1) A_l R^{l-1} P_l(\cos\theta) = \sigma_0(\theta)/\varepsilon_0$$

$$\int_0^\pi \sigma_0(\theta) P_m(\cos\theta) \sin\theta \, d\theta$$

$$= (2m+1) \cdot \varepsilon_0 \cdot A_m \cdot \frac{2}{2m+1} \cdot R^{l-1}$$

$$\Rightarrow A_m = \frac{1}{2 \varepsilon_0 R^{l-1}} \int_0^\pi \sigma_0(\theta) P_m(\cos\theta) \sin\theta \, d\theta$$

If $\sigma_0(\theta) = \text{const.} = \sigma_0$

only $m = 0$ contributes

$$A_0 = \frac{1}{2\varepsilon_0 R^{-1}} \cdot \sigma_0 \cdot \frac{2}{2\cdot 0+1}$$

$$= \frac{\sigma_0 R}{\varepsilon_0}$$

$$\Rightarrow V_<(r,\theta) = \frac{\sigma_0 R}{\varepsilon_0}$$

$$B_0 = A_0 \cdot R^{2\cdot 0+1} = A_0 R$$

$$\Rightarrow V_>(r,\theta) = \frac{\sigma_0 R^2}{\varepsilon_0 r}$$

$$Q = \sigma_0 \cdot 4\pi R^2$$

$$\Rightarrow V_<(r,\theta) = Q/4\pi\varepsilon_0 R$$

$$V_>(r,\theta) = Q/4\pi\varepsilon_0 r \qquad //$$