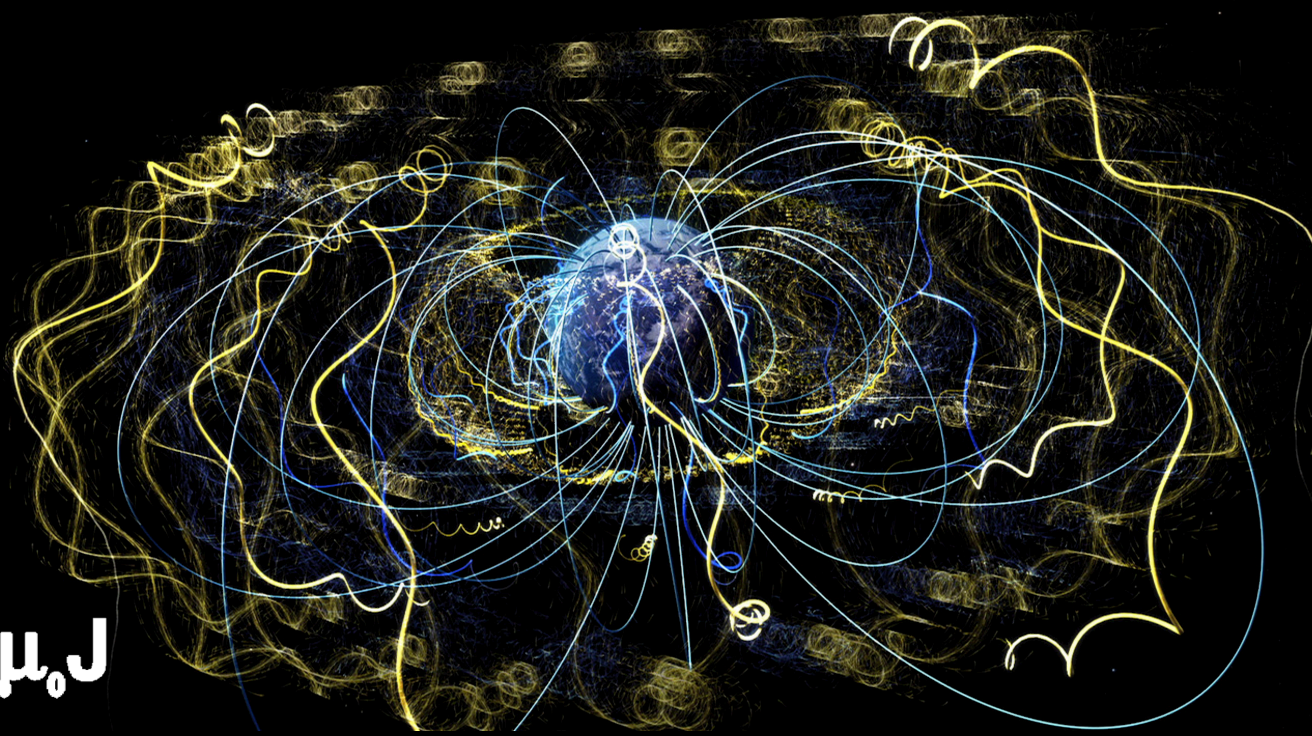
$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}$$



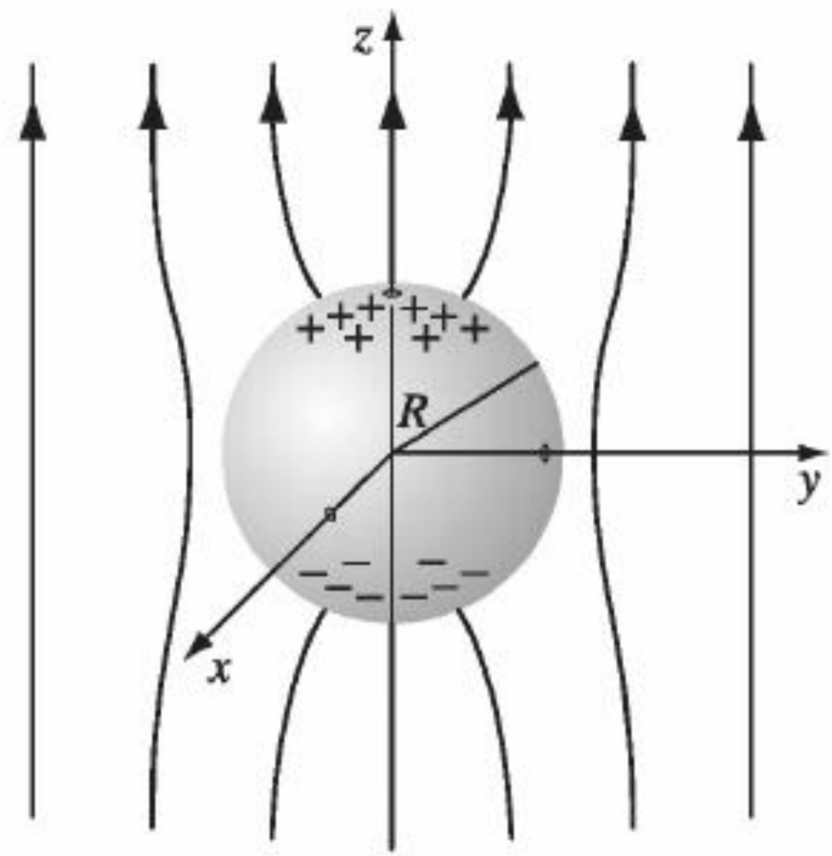
Electricity and Magnetism I: 3811

Professor Jasper Halekas

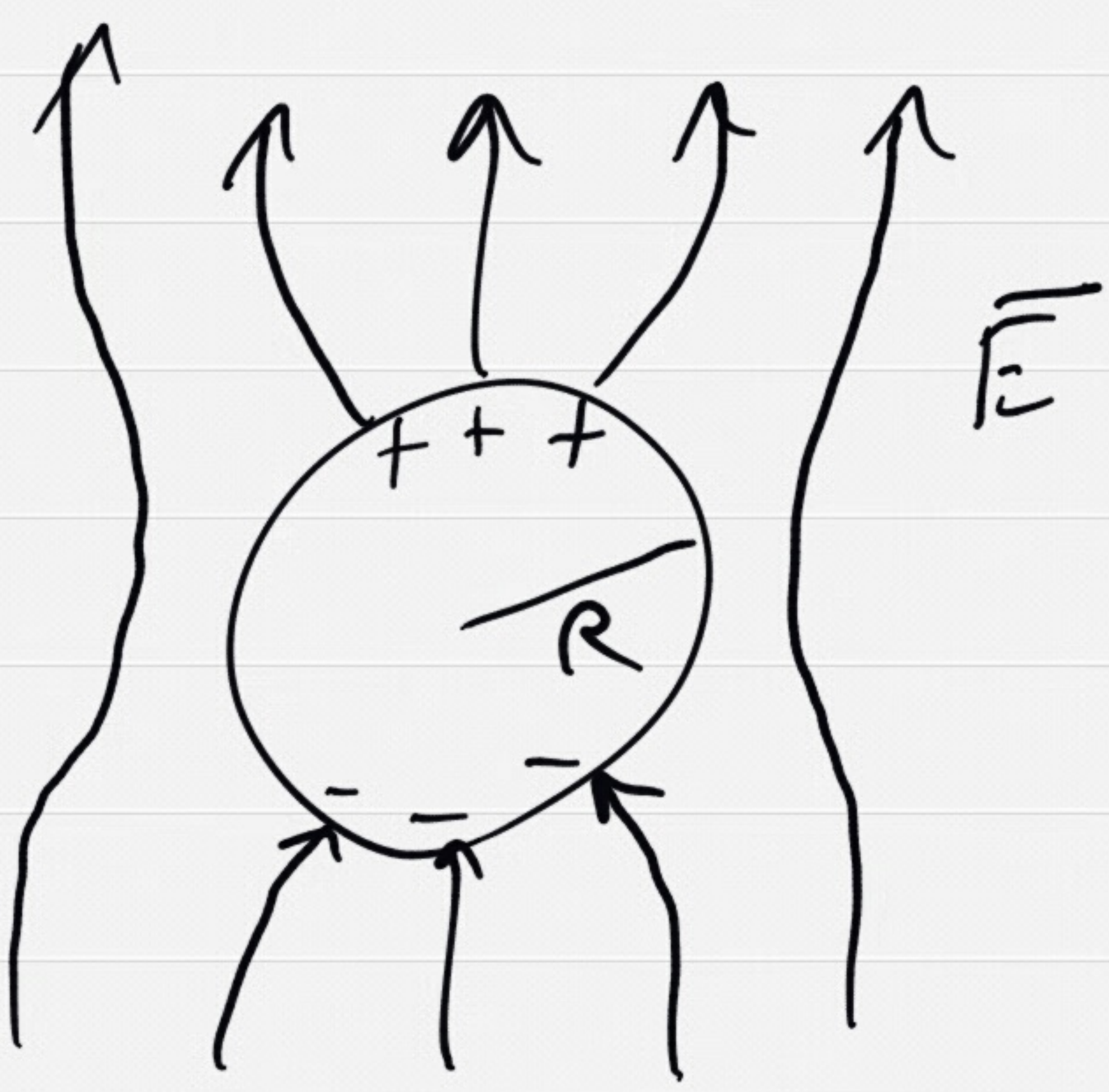
Van Allen 301

MWF 9:30-10:20 Lecture

Conducting Sphere in External Field



Conductor in External Field



$$V = 0 \quad @ \quad r = R$$

$$\vec{E} \to E_0 \hat{z} \quad \text{for} \quad r \gg R$$

$$\Rightarrow V \to -E_0 z$$

$$= -E_0 r \cos\theta$$

$$V(R, \theta) = \sum_\ell \left(A_\ell R^\ell + B_\ell / R^{\ell+1} \right) P_\ell(\cos\theta)$$

$$= 0$$

$$\Rightarrow B_\ell = -A_\ell R^{2\ell+1}$$

$$\Rightarrow V(r, \theta) = \sum_\ell \left(A_\ell r^\ell - \frac{A_\ell R^{2\ell+1}}{r^{\ell+1}} \right) P_\ell(\cos\theta)$$

$$V(r \to \infty, \theta) = \sum_\ell A_\ell r^\ell P_\ell(\cos\theta)$$

$$= -E_0 r \cos\theta$$

$\Rightarrow A_l = 0$ except $l = 1$

$P_1(\cos\theta) = \cos\theta$

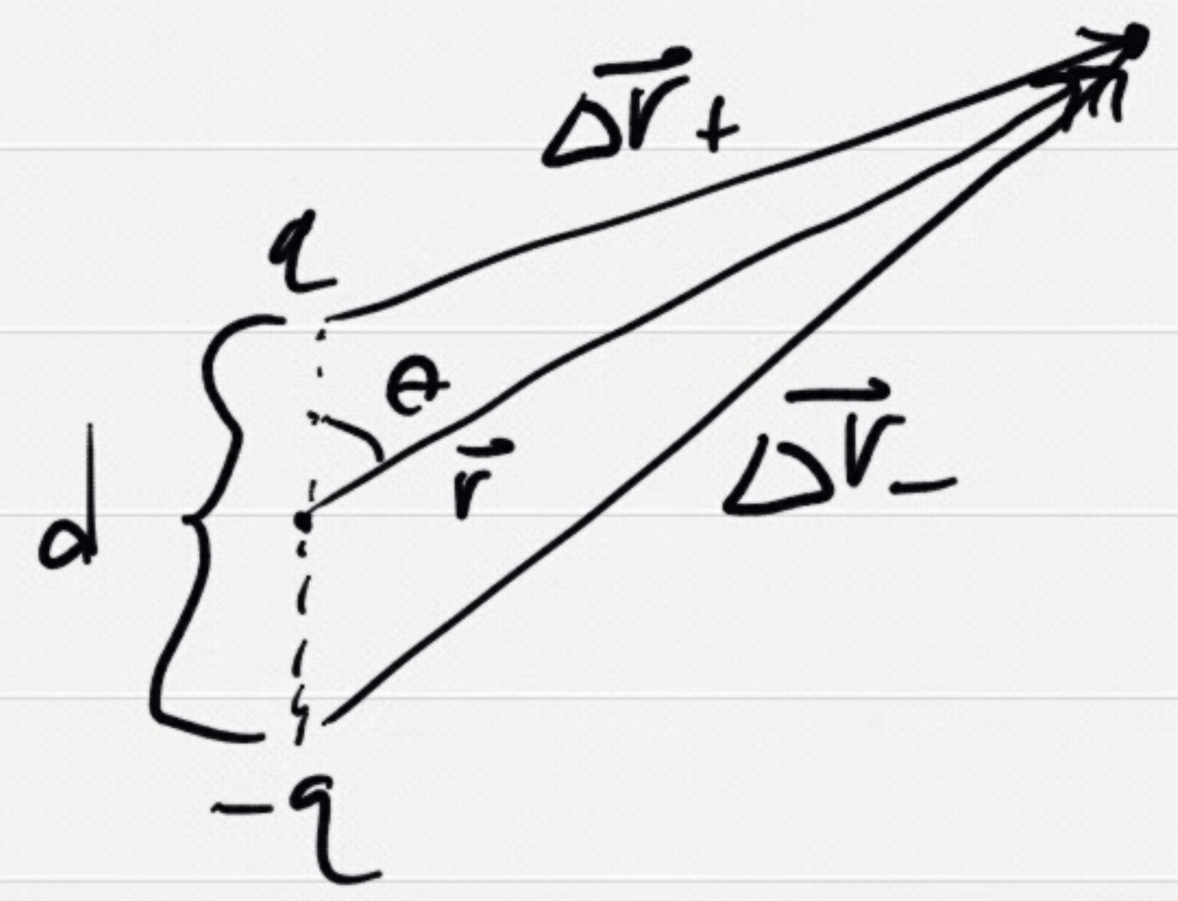
$\Rightarrow A_1 = -E_0$

$$\Rightarrow V(r,\theta) = -E_0\left(r - R^3/r^2\right)\cos\theta$$

$$= -E_0 r\cos\theta + \frac{E_0 R^3}{r^2}\cos\theta$$

= External field + Induced dipole field

Dipole



$$V(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\Delta r_+} - \frac{1}{\Delta r_-}\right)$$

$$\Delta r_\pm^2 = r^2 + \left(d/2\right)^2 \mp 2 \cdot d/2 \cdot r \cdot \cos\theta$$
$$= r^2\left(1 \mp \frac{d}{r}\cos\theta + d^2/4r^2\right)$$

For $d \ll r$

$$\Delta r_\pm \simeq r\sqrt{1 \mp \frac{d}{r}\cos\theta}$$

$$1/\Delta r_\pm \simeq \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right)$$

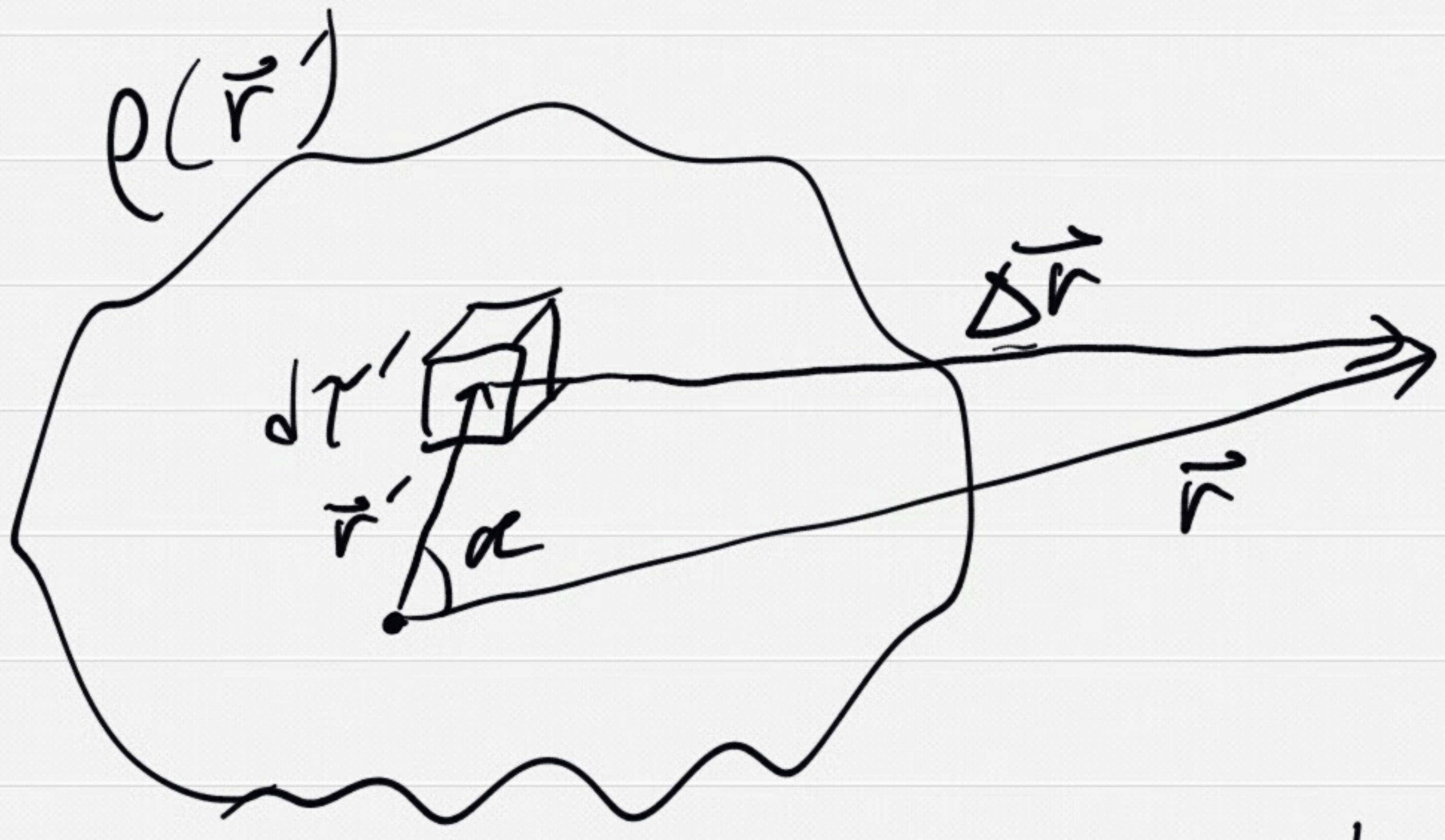
$$\Rightarrow V(\vec{r}) \simeq \frac{q}{4\pi\varepsilon_0} \cdot \frac{1}{r}\left(1 + \frac{d}{2r}\cos\theta - \left(1 - \frac{d}{2r}\cos\theta\right)\right)$$

$$= \frac{q}{4\pi\varepsilon_0} \cdot \frac{d\cos\theta}{r^2}$$

$$= \frac{p\cos\theta}{4\pi\varepsilon_0 r^2} \quad w/\ p = qd$$

Only function of $\vec{r}$, not $\vec{r}_\pm$

Multipole Expansion



$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}')}{\Delta r} d\tau'$$

$$\Delta r^2 = r^2 + r'^2 - 2r'r\cos\alpha$$
$$= r^2\left(1 + \left(\frac{r'}{r}\right)^2 - \frac{2r'}{r}\cos\alpha\right)$$
$$\Delta r = r\sqrt{1 + \frac{r'}{r}\left(\frac{r'}{r} - 2\cos\alpha\right)}$$

Binomial Expansion

$$\frac{1}{\Delta r} = \frac{1}{r}\left(1 + \frac{r'}{r}\left(\frac{r'}{r} - 2\cos\alpha\right)\right)^{-1/2}$$
$$= \frac{1}{r}\left(1 - \frac{1}{2}\frac{r'}{r}\left(\frac{r'}{r} - 2\cos\alpha\right) + \frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\frac{r'}{r} - 2\cos\alpha\right)^2 \cdots\right)$$
$$= \frac{1}{r}\left[1 + \frac{r'}{r}\cos\alpha + \left(\frac{r'}{r}\right)^2\left(\frac{3\cos^2\alpha - 1}{2}\right)\cdots\right]$$

surprisingly $\displaystyle \frac{1}{\Delta r} = \frac{1}{r}\sum_0^\infty \left(\frac{r'}{r}\right)^n P_n(\cos\alpha)$

Legendre Polynomials Again!

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_0^\infty \underbrace{\frac{1}{r^{n+1}}}_{\text{Distance dependence}} \underbrace{\int (r')^n P_n(\cos\alpha)\, \rho(\vec{r}')\, d\tau'}_{\text{All } r'\text{-dependence}}$$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \Big[\frac{1}{r} \int \rho(\vec{r}')\, d\tau'$$
$$+ \frac{1}{r^2} \int r' \cos\alpha\, \rho(\vec{r}')\, d\tau'$$
$$+ \frac{1}{r^3} \int (r')^2 \frac{3\cos^2\alpha - 1}{2} \rho(\vec{r}')\, d\tau'$$
$$- - - - - \Big]$$

Monopole: $V_{mon}(\vec{r}) = \frac{1}{4\pi\varepsilon_0 r} \int \rho(\vec{r}')\, d\tau'$

$$= Q / 4\pi\varepsilon_0 r$$

Dipole: $V_{dip}(\vec{r}) = \frac{1}{4\pi\varepsilon_0 r^2} \int r' \cos\alpha\, \rho(\vec{r}')\, d\tau'$

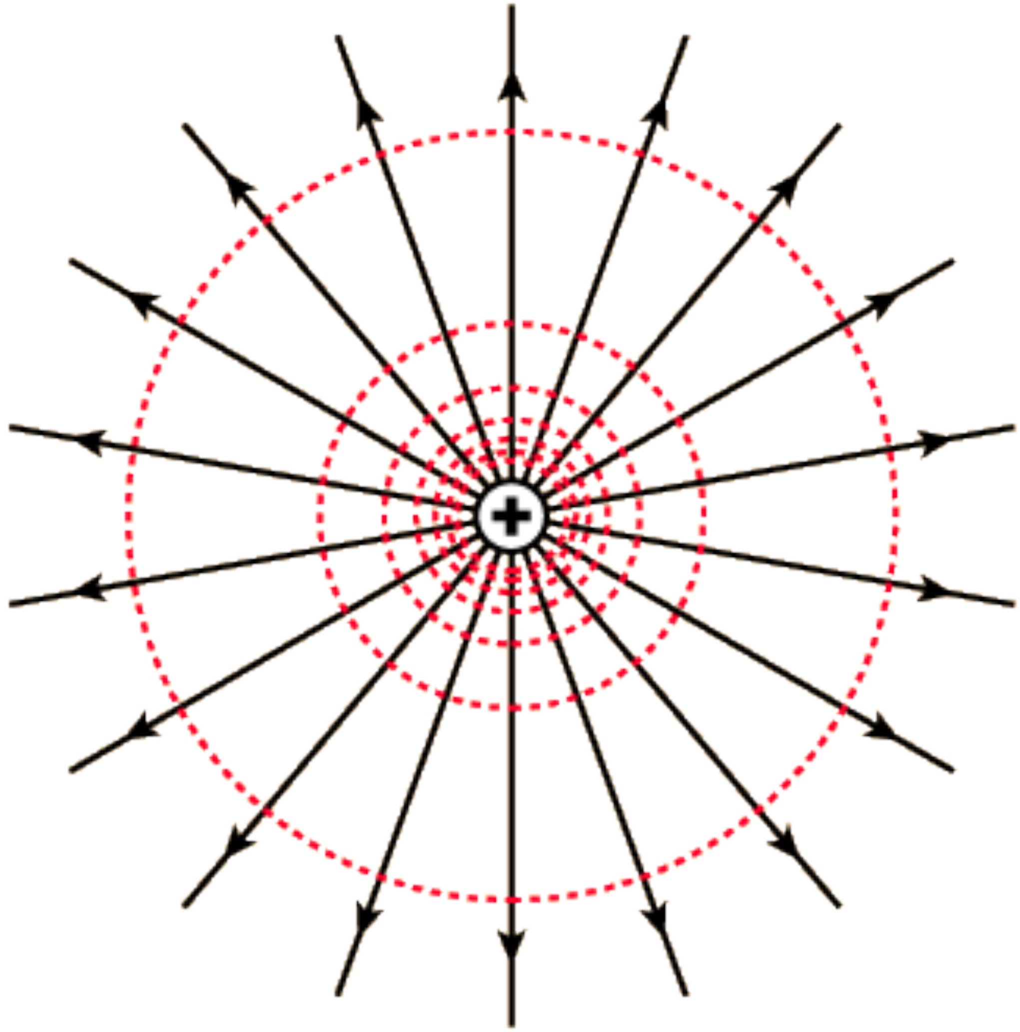
But $r' \cos\alpha = \hat{r} \cdot \vec{r}'$

$$\Rightarrow V_{dip}(\vec{r}) = \frac{1}{4\pi\varepsilon_0 r^2} \hat{r} \cdot \left(\int \vec{r}' \rho(\vec{r}')\, d\tau' \right)$$

$$= \frac{1}{4\pi\varepsilon_0 r^2} \hat{r} \cdot \vec{p}$$

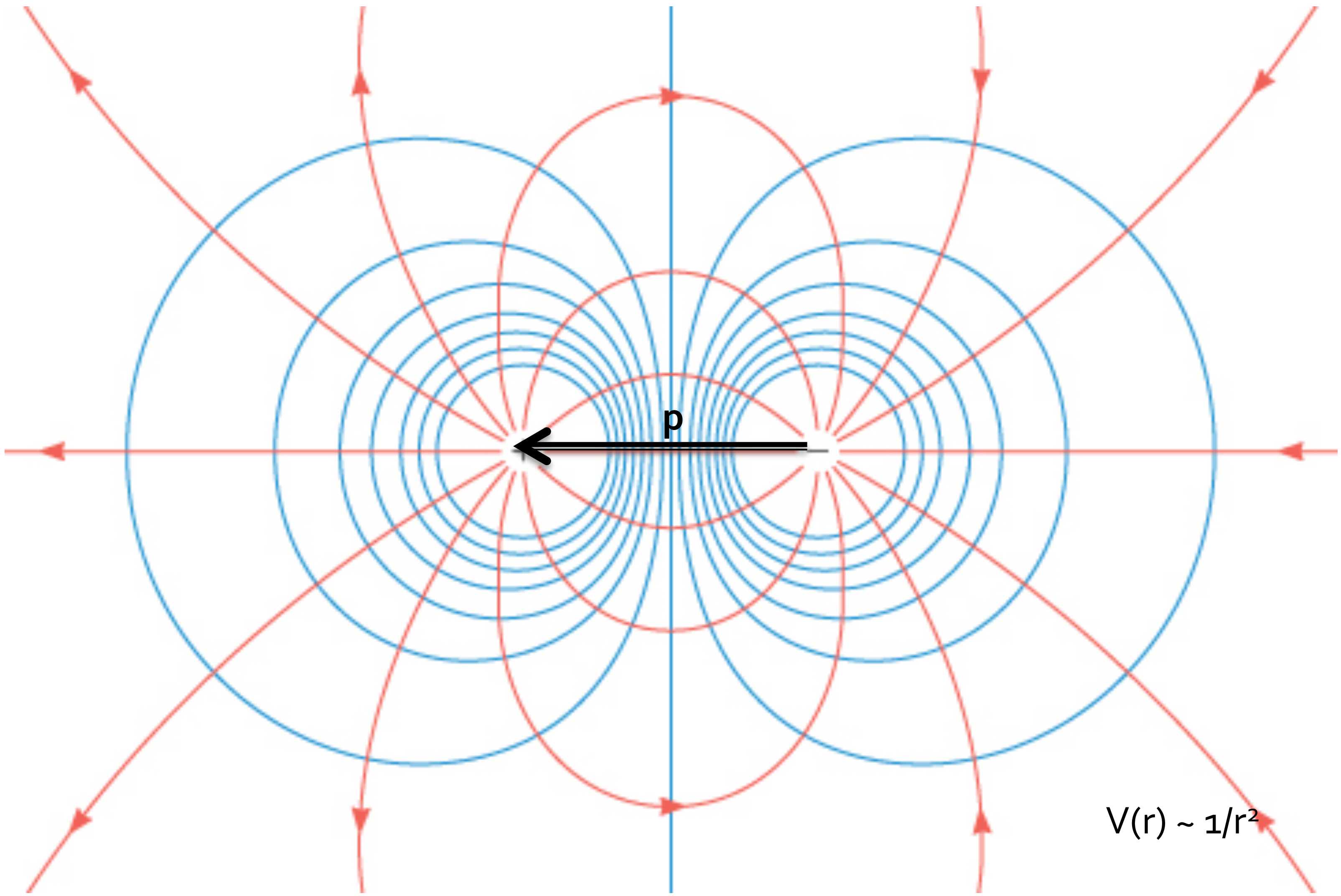
w/ $\vec{p} = \int \vec{r}' \rho(\vec{r}')\, d\tau'$ = dipole moment

Electric Monopole

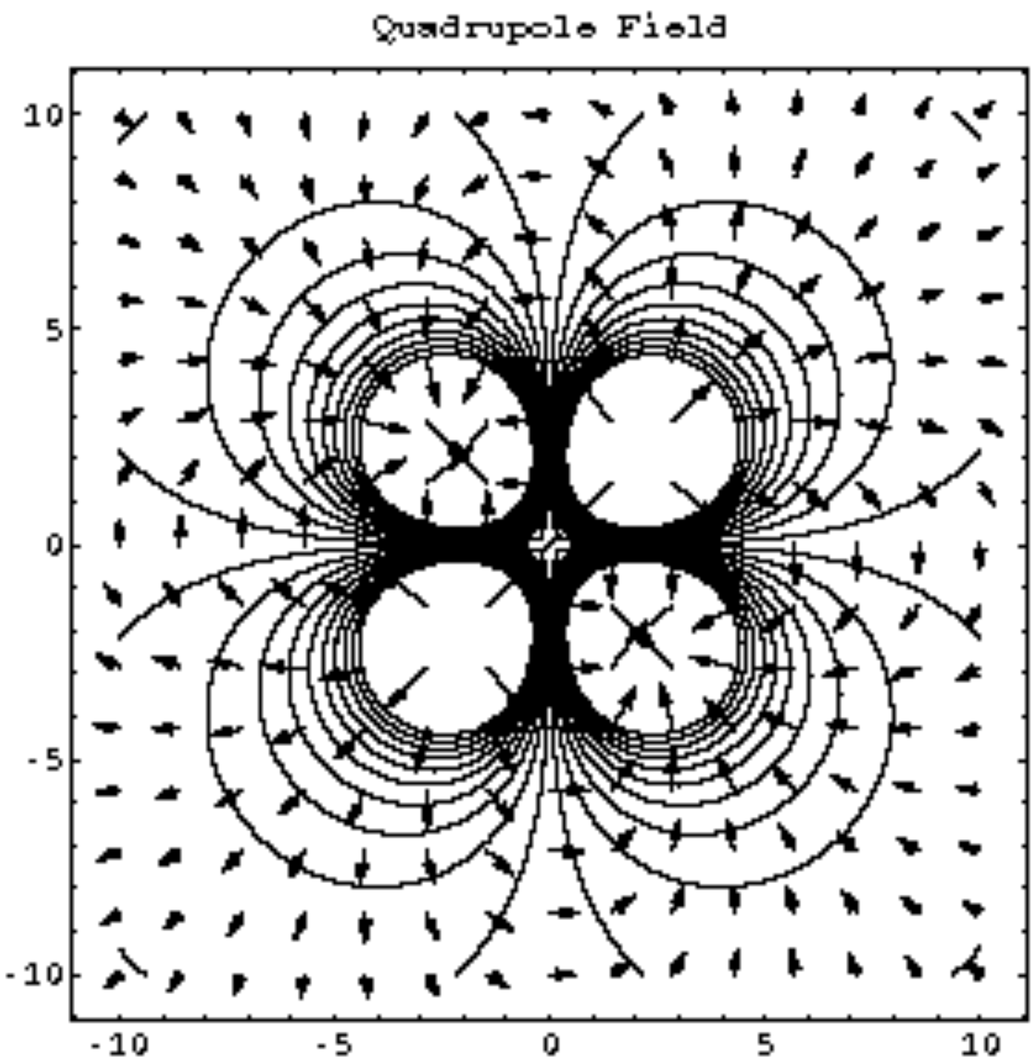


$V(r) \sim 1/r$

Electric Dipole



Electric Quadrupole



$V(r) \sim 1/r^3$