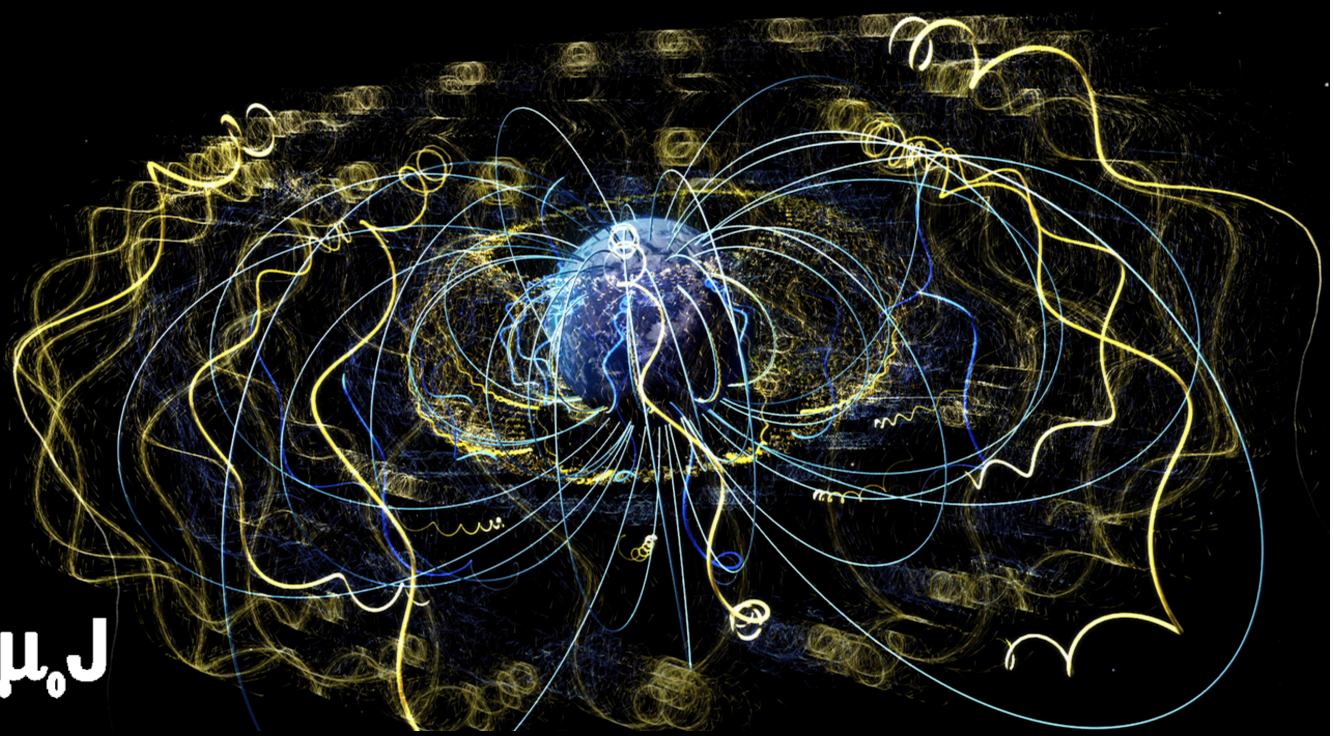
$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

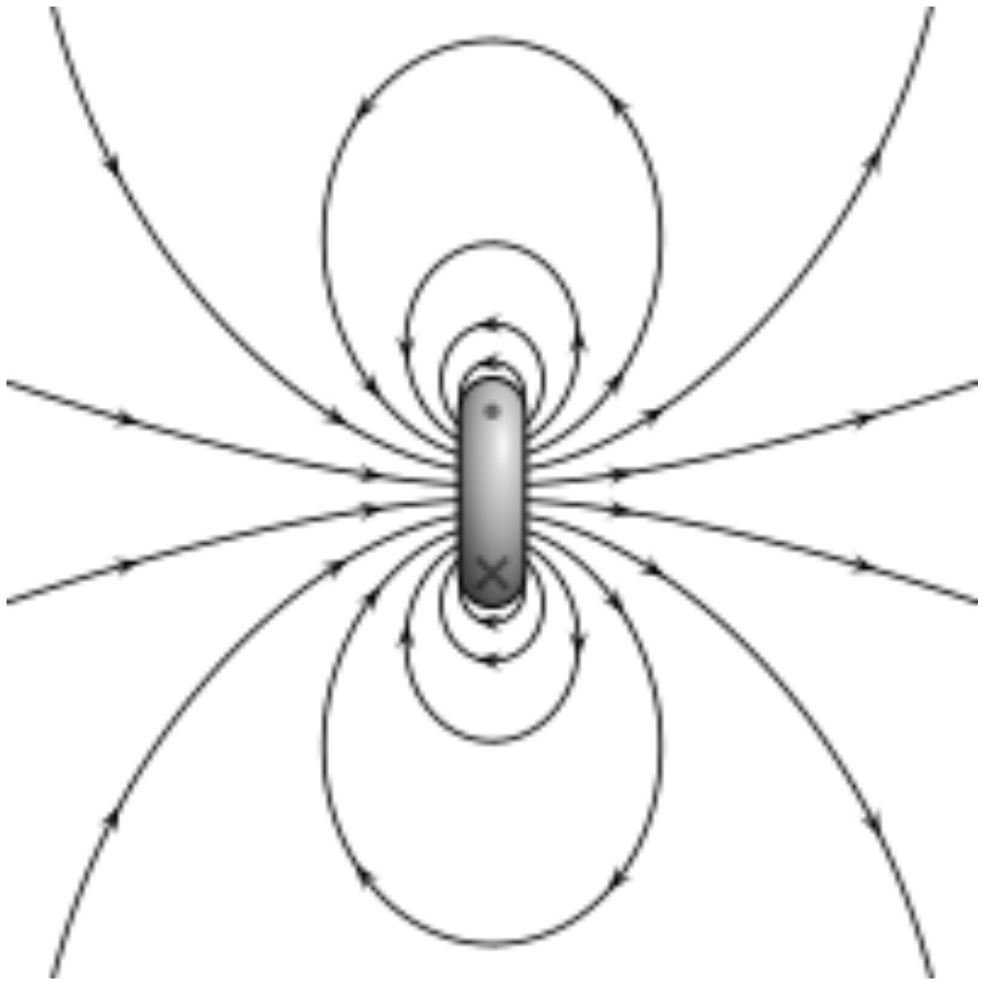
$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}$$



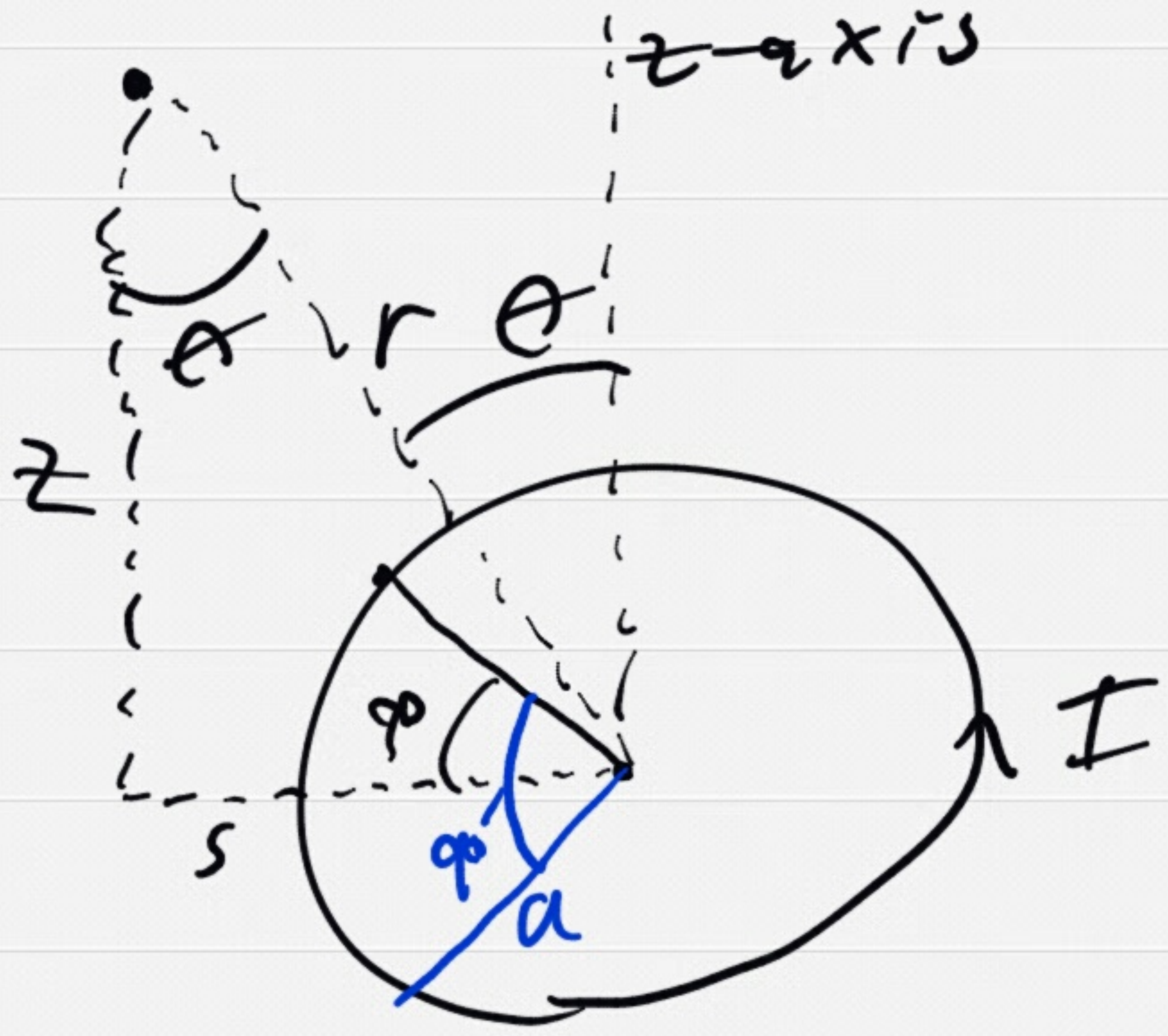
Electricity and Magnetism I: 3811

Professor Jasper Halekas
Van Allen 301
MWF 9:30-10:20 Lecture

Current Loop



Magnetic Vector Potential of Current Loop



$$\vec{r} = (s\cos\varphi, s\sin\varphi, z)$$
$$\vec{r}' = (a\cos\varphi', a\sin\varphi', 0)$$

Pick $\varphi = 0$ for simplicity
(solution will be azimuthally symmetric)

$$\Rightarrow \vec{r} = (s, 0, z)$$

$$\Delta\vec{r} = (s - a\cos\varphi', -a\sin\varphi', z)$$

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l}'}{\Delta r}$$

$$d\vec{l}' = (-a\sin\varphi', a\cos\varphi', 0)\, d\varphi'$$

$$\Delta r = \sqrt{(s - a\cos\varphi')^2 + (a\sin\varphi')^2 + z^2}$$
$$= \sqrt{s^2 - 2as\cos\varphi' + a^2\cos^2\varphi'^2 + a^2\sin^2\varphi'^2 + z^2}$$
$$= \sqrt{s^2 + a^2 + z^2 - 2as\cos\varphi'}$$
$$= \sqrt{r^2 + a^2 - 2as\cos\varphi'}$$

Δr even in φ'
$\sin\varphi'$ odd in φ'

$$\Rightarrow A_x = 0$$

$$A_y = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{a \cos\varphi' \, d\varphi'}{\sqrt{r^2 + a^2 - 2as\cos\varphi'}}$$

For $a << r$

$$\frac{1}{\sqrt{r^2 + a^2 - 2as\cos\varphi'}} \sim \frac{1}{r}\left(1 + \frac{as\cos\varphi'}{r^2}\right)$$

$$\Rightarrow A_y \cong \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \left[\frac{a\cos\varphi'}{r} + \frac{a^2 s \cos^2\varphi'}{r^3}\right] d\varphi'$$

$$= \frac{\mu_0 I}{4\pi} \cdot \frac{a^2 s}{r^3} \cdot \pi$$

but $s/r = \sin\theta$

$$\Rightarrow A_y \cong \frac{\mu_0 I \cdot \pi a^2 \sin\theta}{4\pi r^2}$$

For $\varphi \neq 0$, $A_y \to A_\varphi$

$$\Rightarrow \vec{A} \cong \frac{\mu_0 I \cdot \pi a^2 \cdot \sin\theta}{4\pi r^2} \hat{\varphi}$$

$$= \boxed{\frac{\mu_0 m \sin\theta \, \hat{\varphi}}{4\pi r^2}}$$

w/ $m = IA$ = magnetic moment

$$\vec{B} = \nabla \times \vec{A}$$

$$B_r = \frac{1}{r\sin\theta} \partial/\partial\theta (\sin\theta A_\varphi)$$

$$= \frac{1}{r\sin\theta} \cdot \frac{\mu_0 m \cdot 2\sin\theta\cos\theta}{4\pi r^2}$$

$$= \frac{\mu_0 m \cos\theta}{2\pi r^3}$$

$$B_\theta = -1/r \, \partial/\partial r (r A_\varphi)$$

$$= -1/r \cdot -\frac{\mu_0 m \sin\theta}{4\pi r^2}$$

$$= \frac{\mu_0 m \sin\theta}{4\pi r^3}$$

$$\boxed{\vec{B}(\vec{r}) = \frac{\mu_0 m}{4\pi r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]}$$

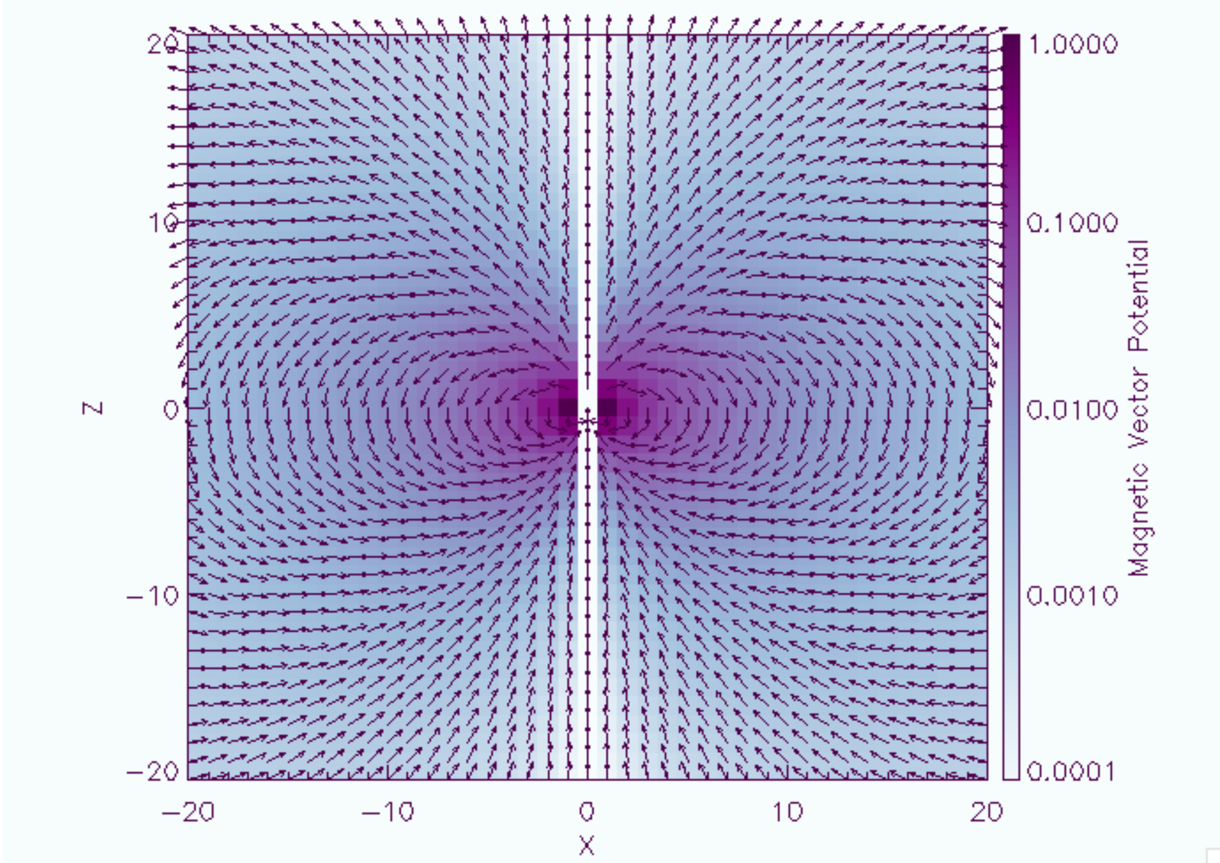
Campare to electric dipole

$$\vec{E}(\vec{r}) = \frac{p}{4\pi\varepsilon_0 r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]$$

A current loop is a magnetic dipole!!

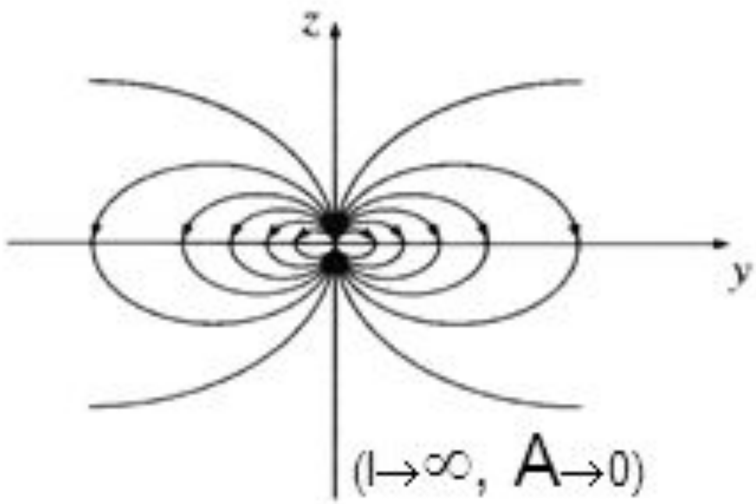
Current Loop = Magnetic Dipole

Color = Magnetic Vector Potential
Vectors = Magnetic Field Direction

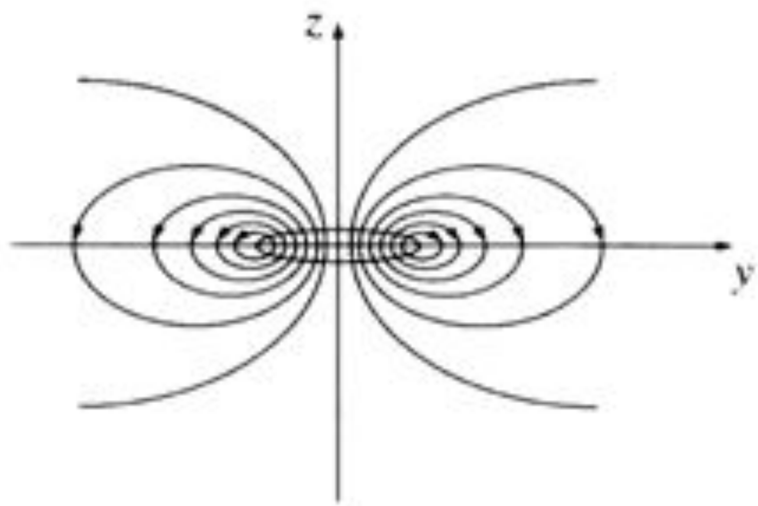


Ideal Vs. Physical Magnetic Dipole

The magnetic field of a magnetic dipole:



Field of a "pure" dipole



Field of a "physical" dipole

Magnetic Vs. Electric Dipole

E-field around electric dipole

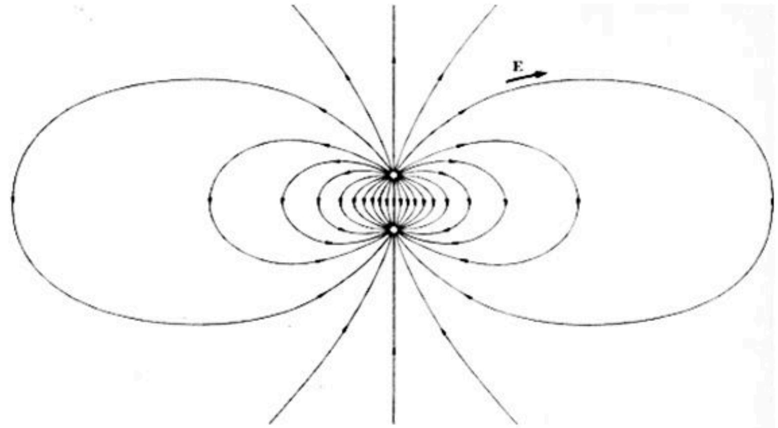
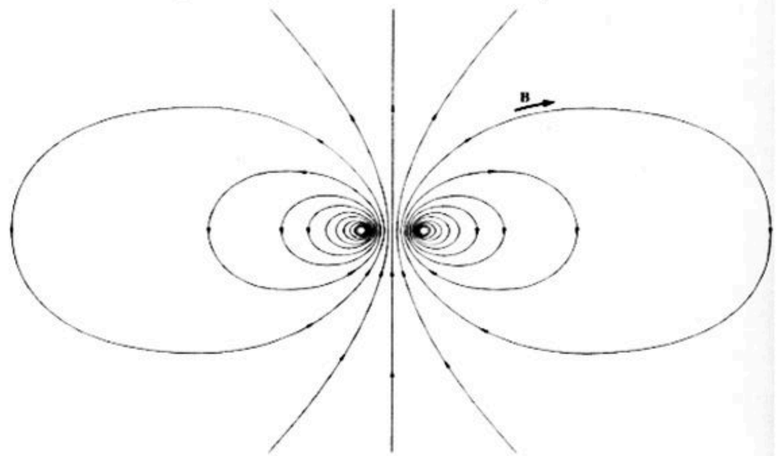


Fig. 10.8 (*a*) The electric field of a pair of equal and opposite charges. Far away it becomes the field of an electric dipole.

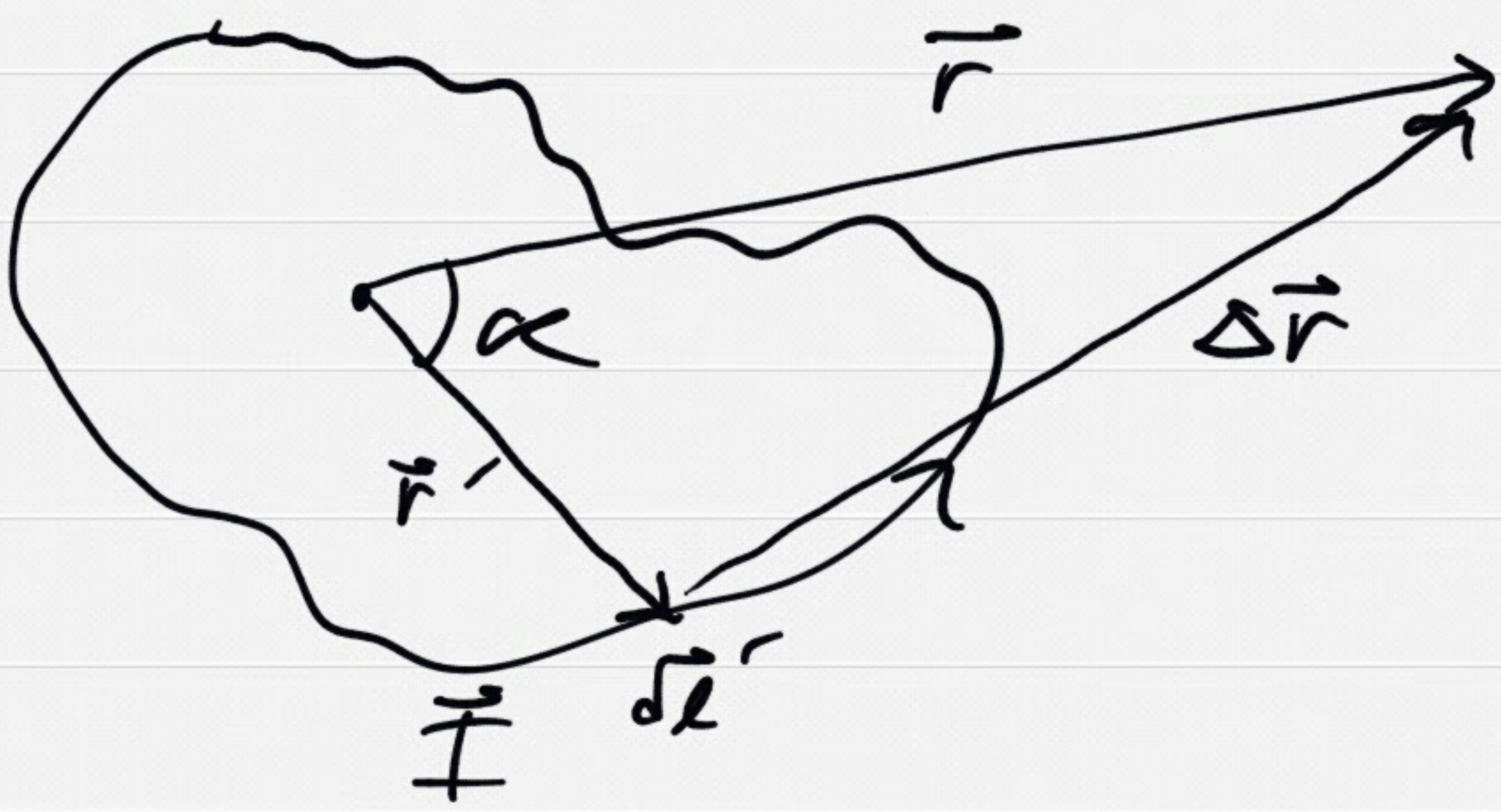
B-field around magnetic dipole (current loop)



(*b*) The magnetic field of a current ring. Far away it becomes the field of a magnetic dipole.

From Purcell,
Electricity and Magnetism

Multipole Expansion



$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l}'}{\Delta r}$$

$$\frac{1}{\Delta r} = \frac{1}{\sqrt{r^2 + (r')^2 - 2rr'\cos\alpha}}$$

$$= \frac{1}{r} \sum_0^\infty \left(\frac{r'}{r}\right)^n P_n(\cos\alpha)$$ as in electrostatics

$$\Rightarrow \vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \sum_0^\infty \frac{1}{r^{n+1}} \oint (r')^n P_n(\cos\alpha)\, d\vec{l}'$$

Monopole Term:

$$\vec{A}_{mon}(\vec{r}) = \frac{\mu_0 I}{4\pi r} \oint d\vec{l}'$$

$$= 0$$ for any closed loop

Dipole Term:

$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0 I}{4\pi r^2} \oint r' \cos\alpha\, d\vec{l}'$$

$$= \frac{\mu_0 I}{4\pi r^2} \oint (\hat{r} \cdot \vec{r}')\, d\vec{l}'$$

$$= \frac{\mu_0 I}{4\pi r^2} \; -\hat{r} \times \int d\vec{a}'$$

So $\vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2}$

w/ $\vec{m} = I \int d\vec{a}'$

= magnetic dipole moment

For flat current loop

$\vec{m} = I \vec{A}$

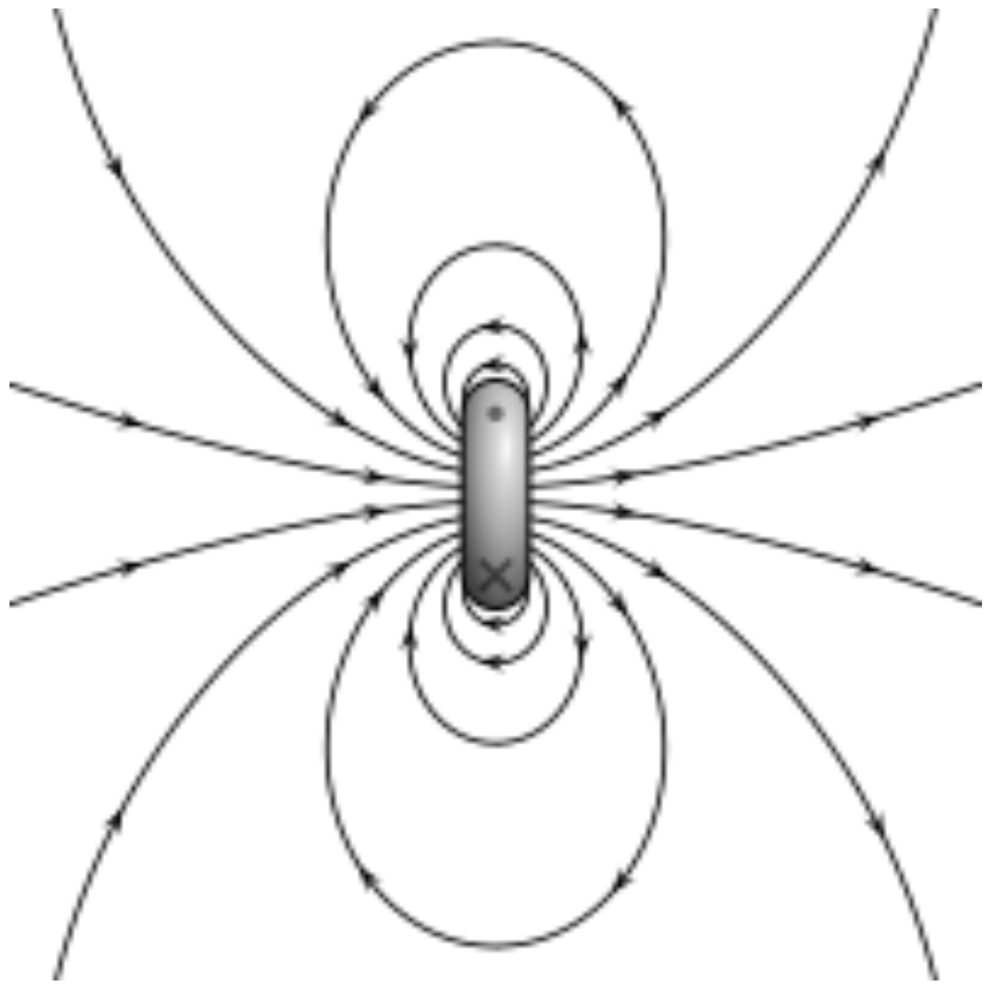
for $\vec{m} = m \hat{z}$

$\vec{A}_{dip} = \frac{\mu_0}{4\pi} \frac{m \sin\theta}{r^2} \hat{\phi}$

- Same as approximate field of current loop for $r \gg a$

Note: Monopole term always cancels even if not a simple current loop

Magnetic Dipole



Magnetic Quadrupole