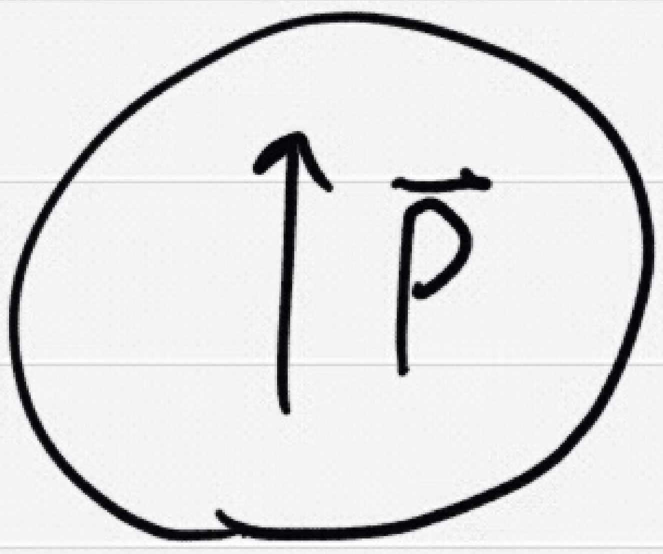
Uniformly Polarized Sphere
(Review)



$\vec{P} = P\,\hat{z}$

Bound charge; $\rho_b = -\nabla \cdot \vec{P} = 0$

$$\sigma_b = \vec{P}\cdot\hat{n} = P\,\hat{z}\cdot\hat{r} = P\cos\theta$$

- Match to separation of variables solution by matching boundary conditions

$$V_<(r,\theta) = \sum A_\ell r^\ell P_\ell(\cos\theta) \qquad r<R$$
$$V_>(r,\theta) = \sum \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta) \qquad r>R$$

$$\Delta V = 0 \Rightarrow V_<(R,\theta) = V_>(R,\theta)$$

$$\Delta \vec{E}_\perp = -\Delta\left(\frac{\partial V}{\partial r}\right)\Big|_R = \frac{\sigma}{\varepsilon_0}$$

$$\Rightarrow V_<(r,\theta) = \frac{P}{3\varepsilon_0} r\cos\theta \qquad r<R$$
$$V_>(r,\theta) = \frac{P}{3\varepsilon_0}\frac{R^3}{r^2}\cos\theta \qquad r>R$$

– Inside : $V_<(r,\theta) = \frac{P}{3\varepsilon_0} r\cos\theta$

$$= \frac{P}{3\varepsilon_0} z$$

$$\Rightarrow \vec{E}_< = -\frac{P}{3\varepsilon_0}\hat{z}$$

$$= -\vec{P}/3\varepsilon_0 \quad \text{(constant!)}$$

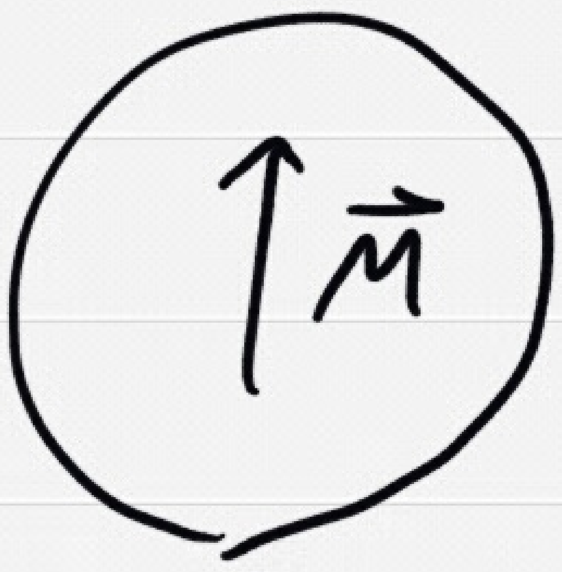
– Outside : $V_>(r,\theta) = \frac{P}{3\varepsilon_0}\frac{R^3}{r^2}\cos\theta$

$$= \frac{4}{3}PR^3 \cdot \frac{\cos\theta}{4\pi\varepsilon_0 r^2}$$

$$= \frac{\vec{p}\cdot\hat{r}}{4\pi\varepsilon_0 r^2}$$

Dipole Potential

Uniformly Magnetized Sphere



$$\vec{M} = M\hat{z}$$

Bound current: $\vec{J}_b = \nabla \times \vec{M} = 0$

$$\vec{K}_b = \vec{M} \times \hat{n}$$
$$= M\,\hat{z} \times \hat{r}$$
$$= M \sin\theta\,\hat{\varphi}$$

- How to solve?
- Could use brute force

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{K}_b(\vec{r}')}{\Delta r}\, da'$$

w/ $\vec{\Delta r} = \vec{r} - \vec{r}'$

- But there are easier ways if you've already solved the electrostatic case

Electrostatic → Magnetostatic I

No free charge

$$\Rightarrow \nabla \times \vec{E} = 0, \quad \nabla \cdot \vec{D} = 0, \quad \varepsilon_0 \vec{E} = \vec{D} - \vec{P}$$

No free current

$$\Rightarrow \nabla \times \vec{H} = 0, \quad \nabla \cdot \vec{B} = 0, \quad \mu_0 \vec{H} = \vec{B} - \mu_0 \vec{M}$$

$$\varepsilon_0 \vec{E} \longleftrightarrow \mu_0 \vec{H}$$
$$\vec{D} \longleftrightarrow \vec{B}$$
$$\vec{P} \longleftrightarrow \mu_0 \vec{M}$$

Uniformly polarized sphere

$$\vec{E} = -\vec{P}/3\varepsilon_0 \quad \text{inside}$$

⇒ Uniformly magnetized sphere

$$\mu_0 \vec{H} = -\mu_0 \vec{M}/3 \quad \text{inside}$$

$$\boxed{\vec{B} = \mu_0 (\vec{M} + \vec{H}) = \frac{2}{3} \mu_0 \vec{M} \quad \text{inside}}$$

Electrostatic → Magnetostatic II

More generally

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{r}\,')\,\hat{\Delta r}}{\Delta r^2}\, d\tau'$$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\vec{P}(\vec{r}\,')\cdot\hat{\Delta r}}{\Delta r^2}\, d\tau'$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{M}(\vec{r}\,')\times\hat{\Delta r}}{\Delta r^2}\, d\tau'$$

For uniform ρ, $\vec{P}$, $\vec{M}$

$$\vec{E}(\vec{r}) = \frac{\rho}{4\pi\varepsilon_0} \int \frac{\hat{\Delta r}}{\Delta r^2}\, d\tau'$$

$$V(\vec{r}) = \frac{\vec{P}}{4\pi\varepsilon_0} \cdot \int \frac{\hat{\Delta r}}{\Delta r^2}\, d\tau'$$

$$A(\vec{r}) = \frac{\mu_0 \vec{M}}{4\pi} \times \int \frac{\hat{\Delta r}}{\Delta r^2}\, d\tau'$$

- Integral part is the same!
- If you've done a given geometry once, you can reuse it

Uniformly polarized sphere

$$V(\vec{r}) = \frac{P}{3\varepsilon_0} r \cos\theta = \frac{1}{3\varepsilon_0} \vec{P} \cdot \vec{r}$$

So $\displaystyle \int \frac{\hat{\Delta r}}{\Delta r^2} \, d\tau' = \frac{4\pi}{3} \vec{r}$ for sphere

$\Rightarrow$ Uniformly magnetized sphere

$$\vec{A}(\vec{r}) = \frac{\mu_0 \vec{M}}{4\pi} \times \frac{4\pi}{3} \vec{r}$$

$$= \frac{\mu_0}{3} \vec{M} \times \vec{r}$$

$$\vec{B}(\vec{r}) = \nabla \times \vec{A}$$

$$= \frac{\mu_0}{3} \nabla \times (\vec{M} \times \vec{r})$$

$$= \frac{\mu_0}{3} \left[(\vec{r} \cdot \nabla)\vec{M} - (\vec{M} \cdot \nabla)\vec{r} + \vec{M}(\nabla \cdot \vec{r}) - \vec{r}(\nabla \cdot \vec{M}) \right]$$

$$= \frac{\mu_0}{3} \left[0 - \vec{M} + 3\vec{M} - 0 \right]$$

$$= \boxed{\frac{2}{3} \mu_0 \vec{M}} \quad //$$