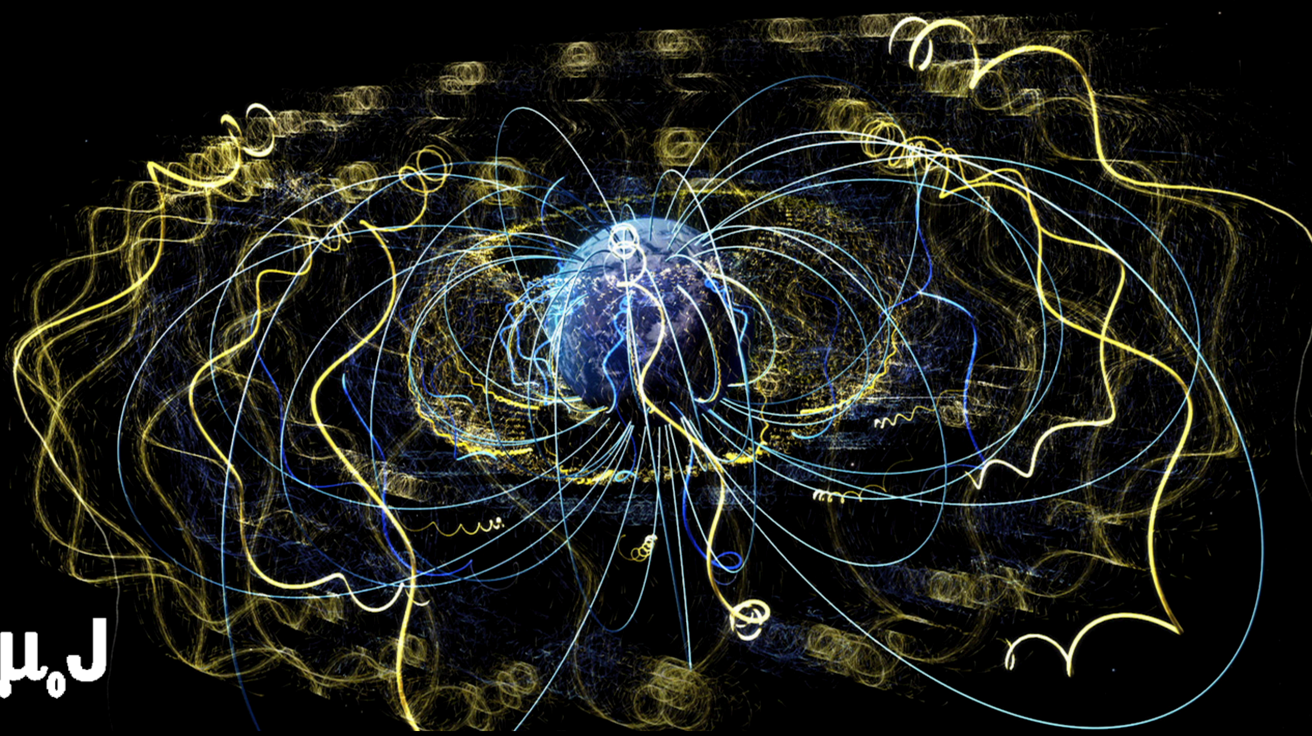
$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}$$



Electricity and Magnetism I: 3811

Professor Jasper Halekas
Van Allen 301

MWF 9:30-10:20 Lecture

Integration by Parts

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$$

$$\Rightarrow \int_a^b \frac{d}{dx}(fg)\,dx = fg\Big|_a^b$$

$$= \int_a^b f\frac{dg}{dx}\,dx + \int_a^b g\frac{df}{dx}\,dx$$

$$\Rightarrow \int_a^b f\frac{dg}{dx}\,dx = fg\Big|_a^b - \int_a^b g\frac{df}{dx}\,dx$$

Similarly: $\nabla\cdot(f\vec{A}) = f\,\nabla\cdot\vec{A} + \vec{A}\cdot\nabla f$

$$\Rightarrow \int f(\nabla\cdot\vec{A})\,d\tau = \oint (f\vec{A})\cdot d\vec{a} - \int \vec{A}\cdot\nabla f\,d\tau$$

And: $\nabla\times(f\vec{A}) = f\,\nabla\times\vec{A} - \vec{A}\times\nabla f$

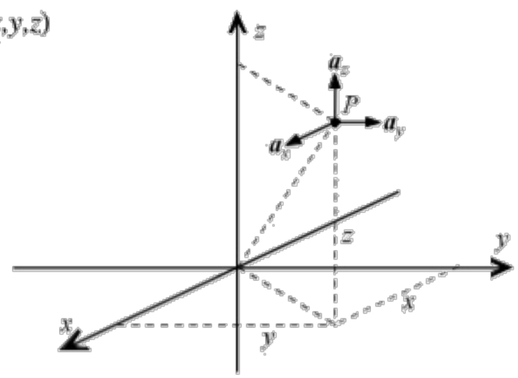
$$\Rightarrow \int f(\nabla\times\vec{A})\cdot d\vec{a} = \oint (f\vec{A})\cdot d\vec{l} + \int (\vec{A}\times\nabla f)\cdot d\vec{a}$$

Coordinate Systems

Coordinate and Unit Vector Definitions

Rectangular Coordinates (x,y,z)

$(-\infty < x < \infty)$
$(-\infty < y < \infty)$
$(-\infty < z < \infty)$

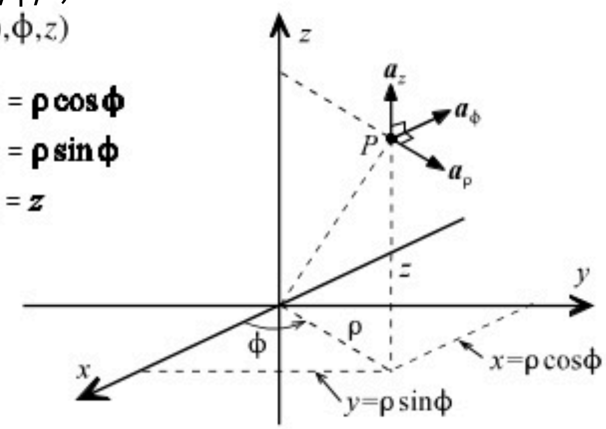


(s,ϕ,z) or

Cylindrical Coordinates (ρ,ϕ,z)

$$s = \rho = \sqrt{x^2+y^2} \qquad x = \rho\cos\phi$$
$$\phi = \tan^{-1}(y/x) \qquad y = \rho\sin\phi$$
$$z = z \qquad z = z$$

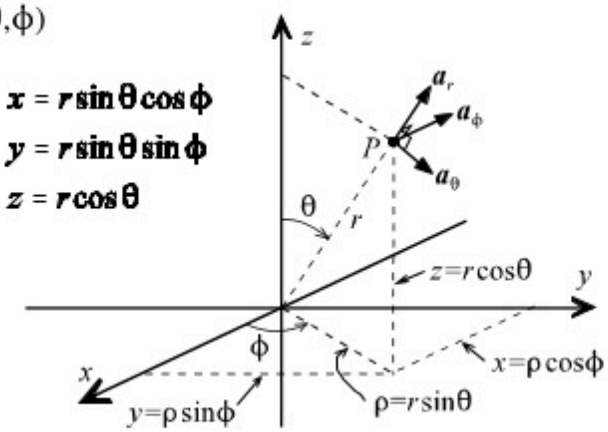
$(0 \le \rho < \infty)$
$(0 \le \phi < 2\pi)$
$(-\infty < z < \infty)$



Spherical Coordinates (r,θ,ϕ)

$$r = \sqrt{x^2+y^2+z^2} \qquad x = r\sin\theta\cos\phi$$
$$\theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right) \qquad y = r\sin\theta\sin\phi$$
$$\phi = \tan^{-1}(y/x) \qquad z = r\cos\theta$$

$(0 \le r < \infty)$
$(0 \le \theta \le \pi)$
$(0 \le \phi < 2\pi)$



Spherical Coordinates

$$\vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$$

or $$\vec{A} = A_r \hat{r} + A_\theta \hat{\theta} + A_\varphi \hat{\varphi}$$

But .... $\hat{x}, \hat{y}, \hat{z}$ always point same direction

$\hat{r}, \hat{\theta}, \hat{\varphi}$ vary w/ position!

Also: dx, dy, dz have units of length, so

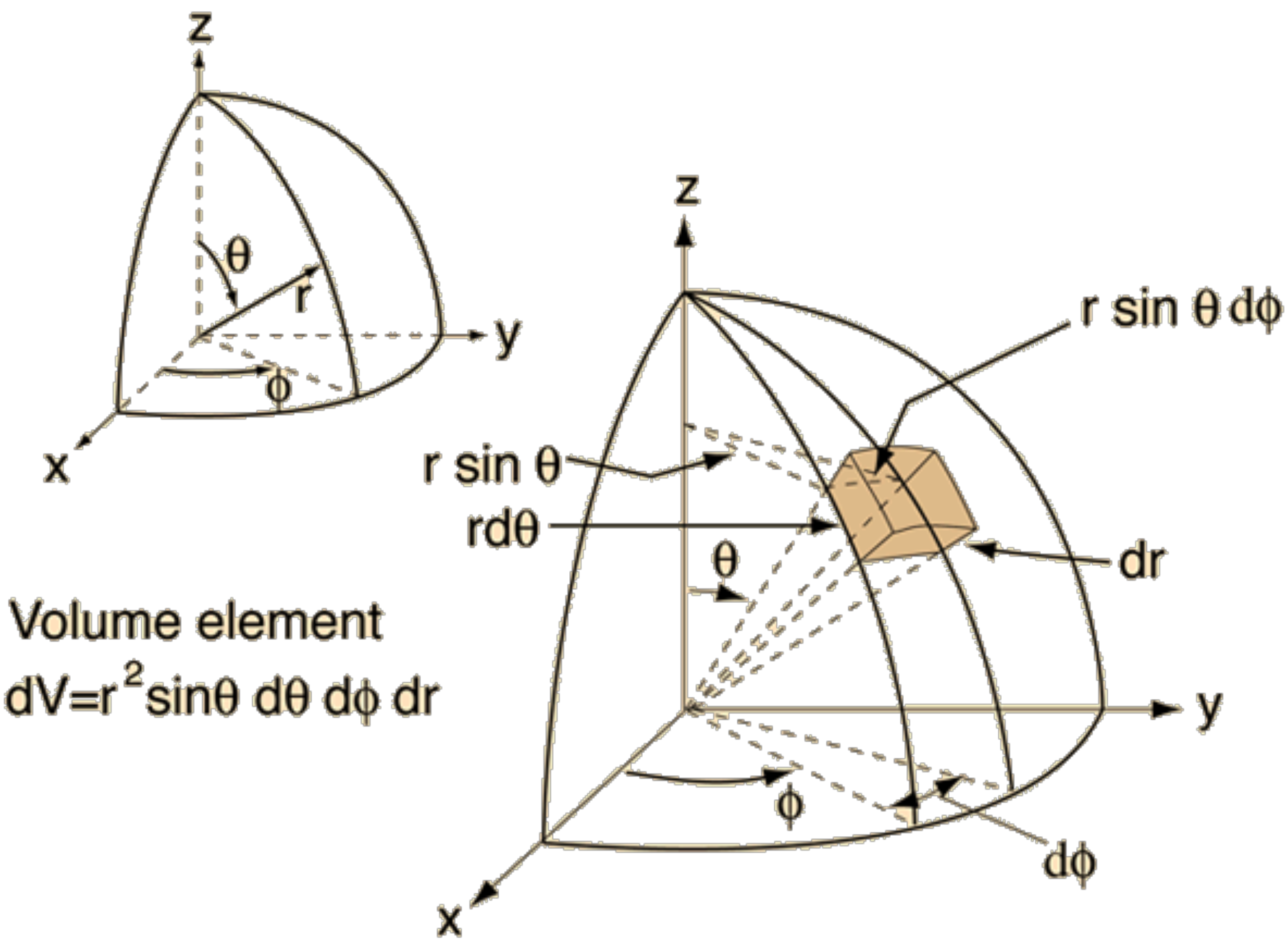
$$d\vec{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

$dr, d\theta, d\varphi$ not all same units

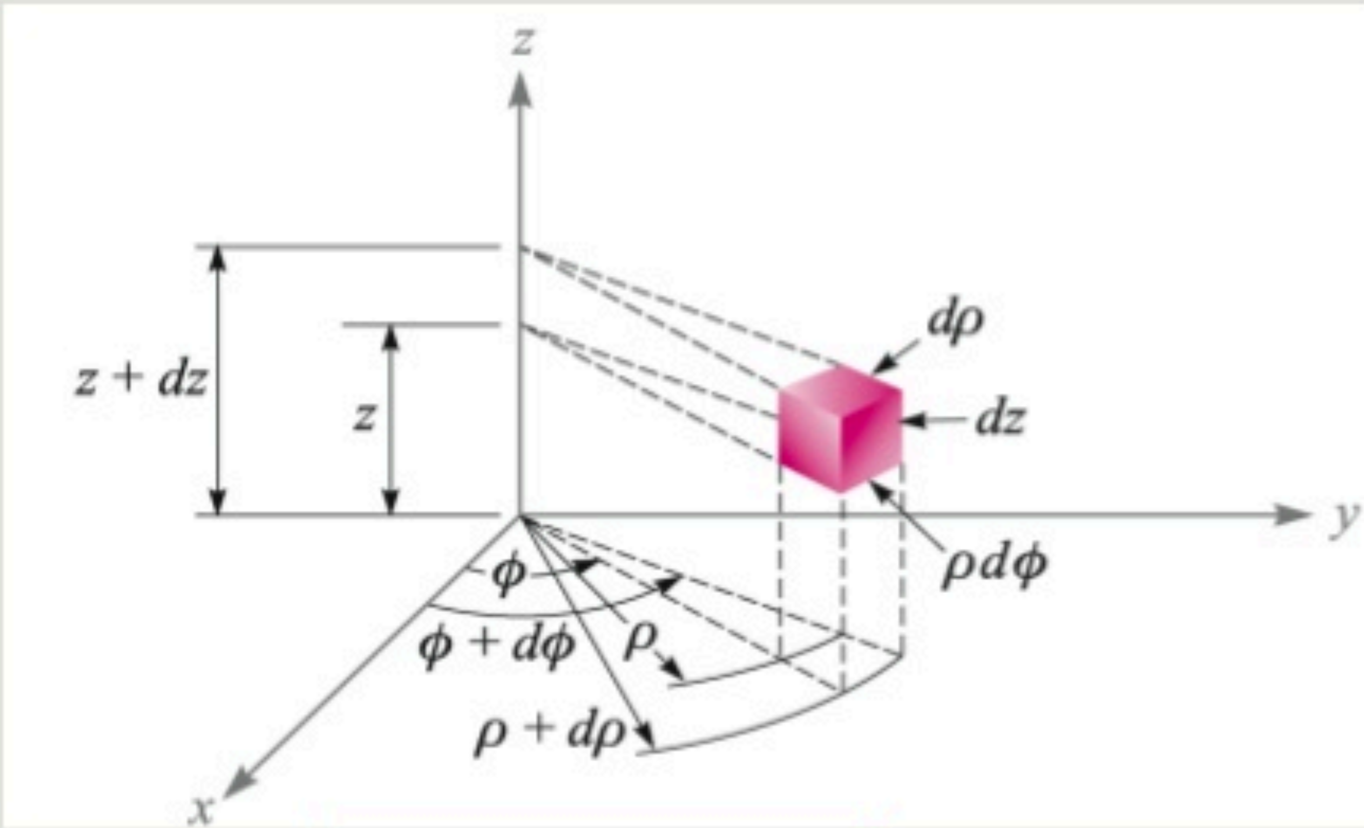
so $$d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\varphi\,\hat{\varphi}$$

Similarly: $$d\tau = dx\,dy\,dz = r^2 \sin\theta\,dr\,d\theta\,d\varphi$$

Spherical Coordinates



Cylindrical Coordinates



$$dV = \rho\, d\rho\, d\phi\, dz$$

$$= s\, ds\, d\varphi\, dz$$

Derivatives in Sphericals

$x = r \sin\theta \cos\varphi \qquad z = r\cos\theta$

$y = r \sin\theta \sin\varphi$

$$\partial/\partial r = \partial x/\partial r \, \partial/\partial x + \partial y/\partial r \, \partial/\partial y + \partial z/\partial r \, \partial/\partial z$$

$$= \sin\theta \cos\varphi \, \partial/\partial x + \sin\theta \sin\varphi \, \partial/\partial y + \cos\theta \, \partial/\partial z$$

$$\hat{r} = \frac{x}{r} \hat{x} + \frac{y}{r} \hat{y} + z/r \, \hat{z}$$

$$= \sin\theta \cos\varphi \, \hat{x} + \sin\theta \sin\varphi \, \hat{y} + \cos\theta \, \hat{z}$$

and similarly for $\partial/\partial\theta$, $\partial/\partial\varphi$ and $\hat{\theta}$, $\hat{\varphi}$

Combine & cancel to get

$$\nabla f = [\partial f/\partial x, \partial f/\partial y, \partial f/\partial z]$$

$$= [\partial f/\partial r, \frac{1}{r} \partial f/\partial\theta, \frac{1}{r\sin\theta} \partial f/\partial\varphi]$$

Derivatives in Spherical & Cylindrical Coordinate Systems

Spherical Coordinates: $x = r\, sin\theta\, cos\phi,\ y = r\, sin\theta\, sin\phi,\ z = r\, cos\theta$

$$\vec{dl} = dr\hat{r} + r\, d\theta\hat{\theta} + r\, sin\theta\, d\phi\hat{\phi} \qquad d\tau = r^2 sin\theta\, dr\, d\theta\, d\phi$$

$$\nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\, sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi} \qquad \nabla \cdot \vec{A} = \frac{1}{r^2}\frac{\partial (r^2 A_r)}{\partial r} + \frac{1}{r\, sin\theta}\frac{\partial (sin\theta\, A_\theta)}{\partial \theta} + \frac{1}{r\, sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\nabla\times\vec{A} = \frac{1}{r\, sin\theta}\left(\frac{\partial (sin\theta\, A_\phi)}{\partial \theta} - \frac{\partial A_\theta}{\partial \phi}\right)\hat{r} + \frac{1}{r}\left(\frac{1}{sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{\partial (r\, A_\phi)}{\partial r}\right)\hat{\theta} + \frac{1}{r}\left(\frac{\partial (r\, A_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right)\hat{\phi}$$

Cylindrical Coordinates: $x = s\, cos\phi,\ y = s\, sin\phi,\ z = z$

$$\vec{dl} = ds\hat{s} + s\, d\phi\hat{\phi} + dz\hat{z} \qquad d\tau = s\, ds\, d\phi\, dz$$

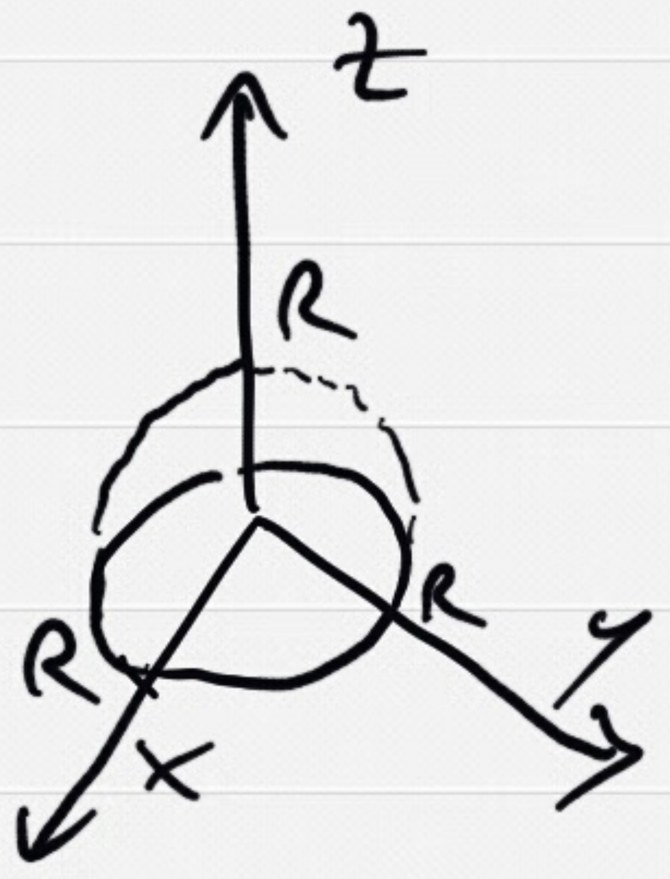
$$\nabla f = \frac{\partial f}{\partial s}\hat{s} + \frac{1}{s}\frac{\partial f}{\partial \phi}\hat{\phi} + \frac{\partial f}{\partial z}\hat{z} \qquad \nabla \cdot \vec{A} = \frac{1}{s}\frac{\partial (s\, A_s)}{\partial s} + \frac{1}{s}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla\times\vec{A} = \left(\frac{1}{s}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{s} + \left(\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s}\right)\hat{\phi} + \frac{1}{s}\left(\frac{\partial (s\, A_\phi)}{\partial s} - \frac{\partial A_s}{\partial \phi}\right)\hat{z}$$

Example:

$$\vec{A} = r\cos\theta\,\hat{r} + r\sin\theta\,\hat{\theta} + r\sin\theta\cos\varphi\,\hat{\varphi}$$

Check divergence theorem for half sphere w/ radius R



$$\nabla\cdot\vec{A} = \frac{1}{r^2}\partial/\partial r\,(r^2 A_r) + \frac{1}{r\sin\theta}\partial/\partial\theta\,(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\partial/\partial\varphi\,A_\varphi$$

$$= \frac{1}{r^2}\partial/\partial r\,(r^3\cos\theta) + \frac{1}{r\sin\theta}\partial/\partial\theta\,(r\sin^2\theta) + \frac{1}{r\sin\theta}\partial/\partial\varphi\,(r\sin\theta\cos\varphi)$$

$$= 3\cos\theta + 2\cos\theta - \sin\varphi$$

$$= 5\cos\theta - \sin\varphi$$

$$\int \nabla\cdot\vec{A}\,d\tau = \int (5\cos\theta - \sin\varphi)\,r^2\,dr\,\sin\theta\,d\theta\,d\varphi$$

$$= \int_0^R \left[\int_0^{\pi/2}\left[\int_0^{2\pi}(5\cos\theta - \sin\varphi)\,d\varphi\right]\sin\theta\,d\theta\right]r^2\,dr$$

$$= \int_0^R \left[\int_0^{\pi/2} 2\pi \cdot 5 \cos\theta \sin\theta \, d\theta \right] r^2 \, dr$$

$$= \int_0^R \left[10\pi \frac{\sin^2\theta}{2} \Big|_0^{\pi/2} \right] r^2 \, dr$$

$$= \int_0^R 5\pi r^2 \, dr = \frac{5}{3}\pi r^3 \Big|_0^R$$

$$= \boxed{5/3 \, \pi R^3}$$

$$\int_{top} \vec{A} \cdot d\vec{a} = \int_{top} \vec{A} \cdot r^2 \sin\theta \, d\theta \, d\varphi \, \hat{r}$$

$$= \int_{top} R^3 \sin\theta \cos\theta \, d\theta \, d\varphi$$

$$= \int_0^{2\pi} \left[\int_0^{\pi/2} R^3 \sin\theta \cos\theta \, d\theta \right] d\varphi$$

$$= \int_0^{2\pi} \left[R^3 \frac{\sin^2\theta}{2} \Big|_0^{\pi/2} \right] d\varphi$$

$$= \int_0^{2\pi} \frac{R^3}{2} \, d\varphi = \pi R^3$$

$$\int_{bottom} \vec{A} \cdot d\vec{a} = \int_{bottom} \vec{A} \cdot r \sin(\pi/2) \, dr \, d\varphi \, \hat{\theta}$$

$$= \int_{bottom} r^2 \sin^2(\pi/2) \, dr \, d\varphi$$

$$= \int_{bottom} r^2 \, dr \, d\varphi$$

$$= \int_0^{2\pi} \left[\int_0^R r^2 \, dr \right] d\varphi$$

$$= \int_0^{2\pi} \left[r^3/3 \Big|_0^R \right] d\varphi$$

$$= \int_0^{2\pi} R^3/3 \, d\varphi = 2\pi R^3/3$$

$$\Rightarrow \oint \vec{A} \cdot d\vec{a} = \boxed{5/3 \, \pi R^3} \quad //$$