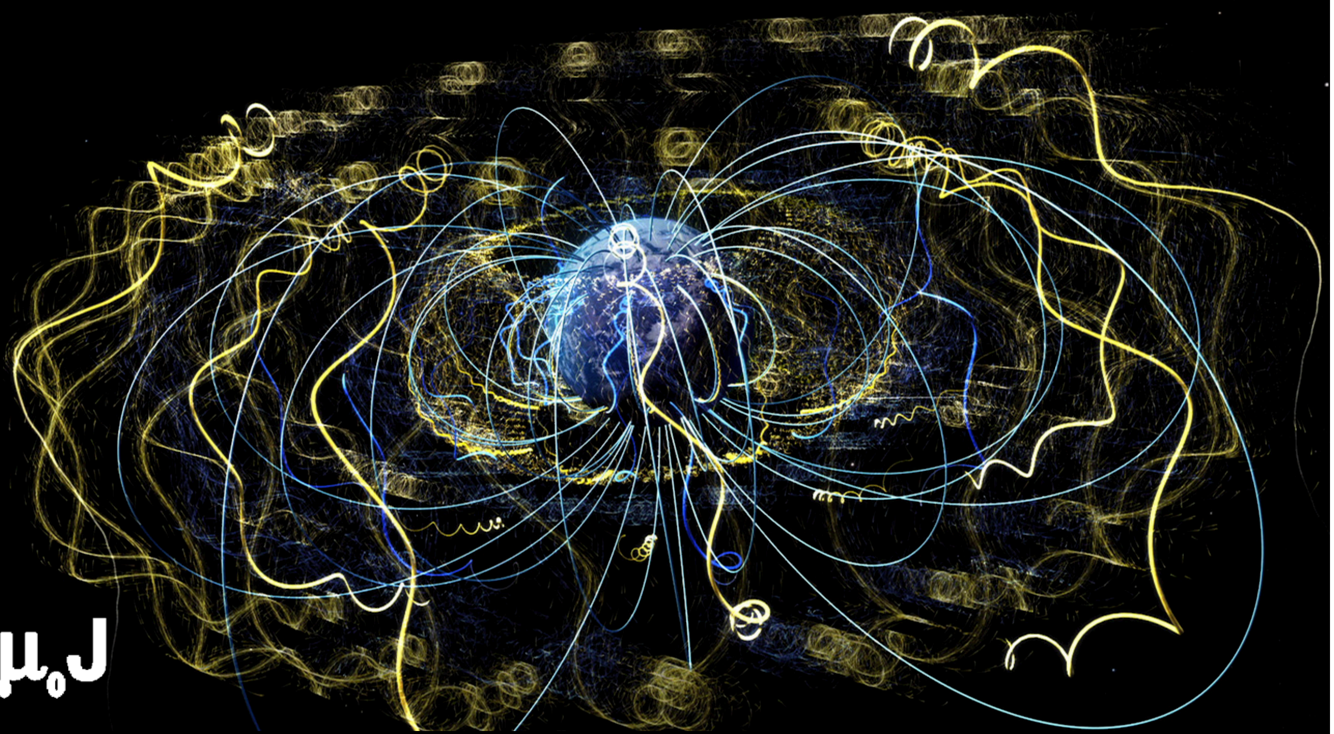
$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}$$



Electricity and Magnetism I: 3811

Professor Jasper Halekas
Van Allen 301
MWF 9:30-10:20 Lecture

Delta Function

1-dimensional Delta Function

$\delta(x)$

Defined by its integral

$$\int_{-\infty}^{\infty} \delta(x)\,dx = 1$$

$$\int_{-\infty}^{a} \delta(x)\,dx = \theta(a)$$

where $\theta(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$

- So $\delta(x)$ is non-zero only at $x = 0$

- The value of $\delta(x)$ at $x = 0$ is infinite!

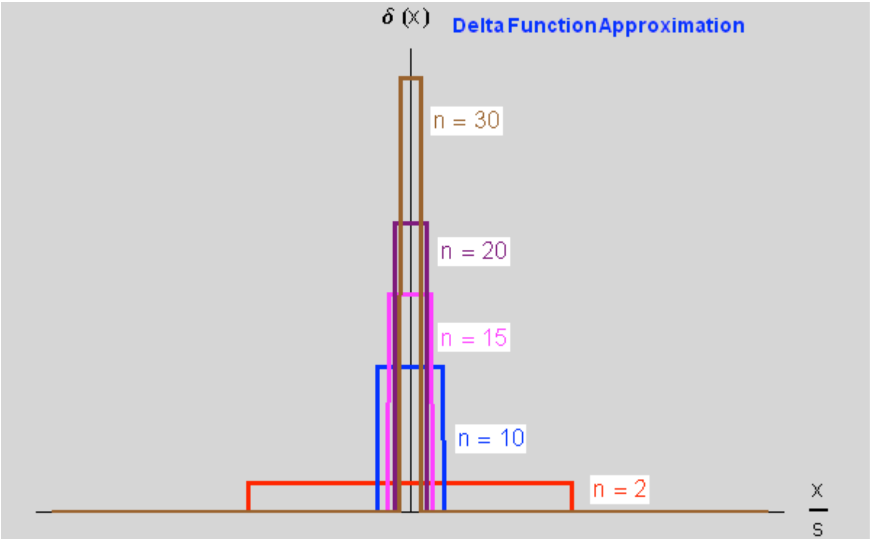
- Since $\delta(x)$ non-zero only at $x=0$

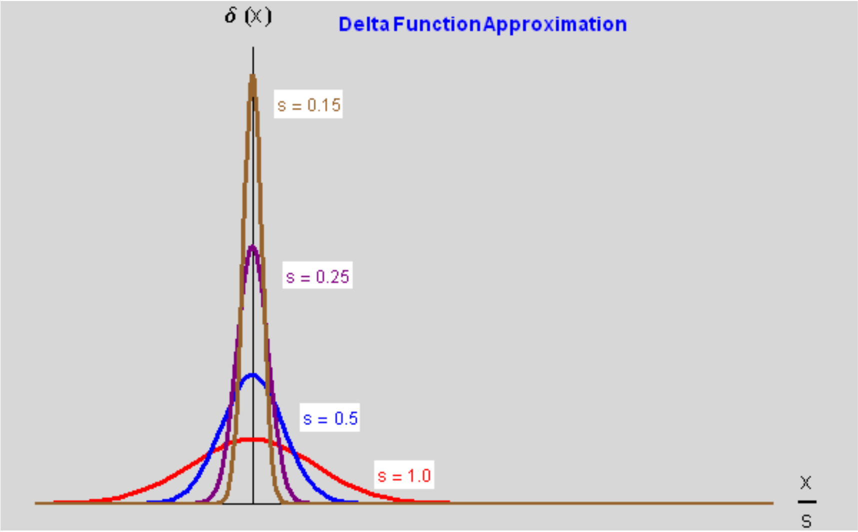
$$f(x)\,\delta(x) = f(0)\,\delta(x)$$

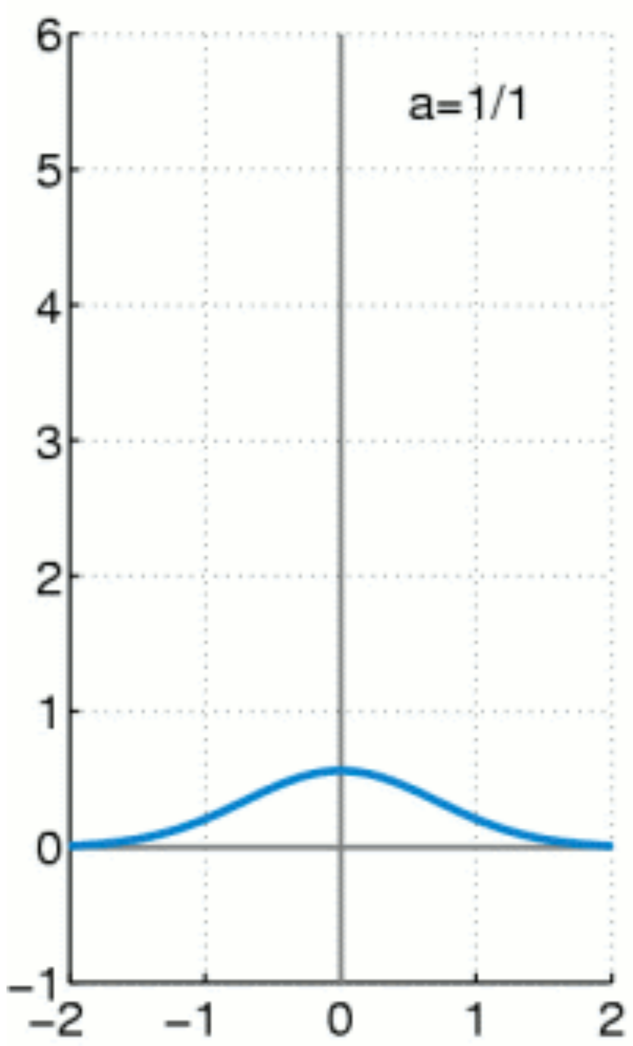
$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = f(0)$$

Delta function picks out value of $f(x)$ at origin

Delta Function Approximations







Shifted Delta Function

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx$$

$$= \int_{-\infty}^{\infty} f(a)\,\delta(x-a)\,dx$$

$$= f(a)$$

Rescaled Delta Function

$$\int_{-\infty}^{\infty} f(x)\,\delta(kx)\,dx$$

put $y = kx$, $dy = k\,dx$

$$\Rightarrow \int_{-\infty}^{\infty} f(y/k)\,\delta(y)\cdot dy/k \quad \text{if } k > 0$$

$$= \int_{\infty}^{-\infty} f(y/k)\,\delta(y)\cdot dy/k \quad \text{if } k < 0$$

$$\Rightarrow \delta(y) = \boxed{\delta(kx) = \frac{1}{|k|}\,\delta(x)}$$

3-d Delta Function

$$\delta^3(\vec{r}) = \delta^3(x, y, z)$$
$$= \delta(x)\,\delta(y)\,\delta(z)$$

So $\int_V \delta^3(\vec{r})\, d\tau = 1$

if V contains the origin

$$\int_V f(\vec{r})\,\delta^3(\vec{r})\, d\tau = f(0)$$

if 0 in V

$$\int_V f(\vec{r})\,\delta^3(\vec{r}-\vec{a})\, d\tau = f(\vec{a})$$

if $\vec{a}$ in V

If $\vec{a} = \vec{r}\,'$, $\vec{r}-\vec{a} = \vec{r}-\vec{r}\,' = \Delta\vec{r}$

$$\int_{\text{all space}} f(\vec{r})\,\delta^3(\Delta\vec{r})\, d\tau = f(\vec{r}\,')$$

$\delta^3(\Delta\vec{r})$ picks out value of $f(\vec{r})$ at source point $\vec{r}\,'$

Real-World Example

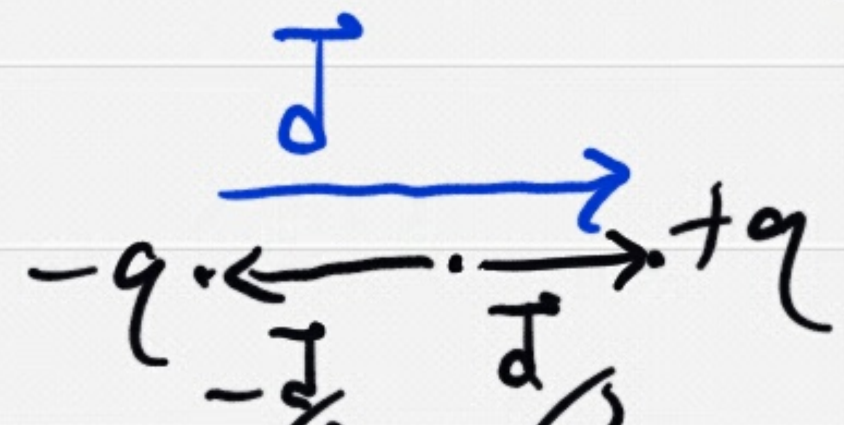
Charge density of electron at point $\vec{r}\,'$

$$\rho(\vec{r}) = q\,\delta^3(\vec{r}-\vec{r}\,')$$
$$= q\,\delta^3(\Delta\vec{r})$$

- Total charge $Q = \int_V \rho(\vec{r})\,d\tau$
$$= q \text{ if } \vec{r}\,' \text{ in } V$$

- For a true point particle, charge density has to be a delta function
  - No charge anywhere except source point $\vec{r}\,'$
  - Total charge $= q$

Dipole:



$$\rho(\vec{r}) = q\,\delta^3(\vec{r}-\vec{d}/2) - q\,\delta^3(\vec{r}+\vec{d}/2)$$

Delta Function Solves a "Paradox"

$$\vec{A} = \hat{r}/r^2$$

$$\nabla \cdot \vec{A} = 1/r^2 \, \partial/\partial r \, (r^2 A_r)$$

$$= 1/r^2 \, \partial/\partial r \left(r^2 \cdot \frac{1}{r^2}\right)$$

$$= 1/r^2 \, \partial/\partial r (1)$$

$$\stackrel{?}{=} 0$$

So $\int_V (\nabla \cdot \vec{A}) \, d\tau = 0$?

for V a sphere of radius R around origin

But $\oint \vec{A} \cdot d\vec{a} = \oint \frac{\hat{r}}{r^2} \cdot \hat{r} \, da$

$$= \oint \frac{1}{r^2} \cdot r^2 \sin\theta \, d\theta \, d\varphi$$

$$= \oint \sin\theta \, d\theta \, d\varphi$$

$= 4\pi$ for any R

What's up!?!

Actually:

$\nabla \cdot \vec{A}$ indeterminate at origin, because

$$\frac{\hat{r}}{r^2} \to \infty \quad @ \quad r=0$$

so $\partial/\partial r\left(r^2 \cdot \frac{1}{r^2}\right)$ indeterminate

Solution:

$$\nabla \cdot \left(\frac{\hat{r}}{r^2}\right) = 4\pi\,\delta^3(\vec{r})$$

so $\int_V \nabla \cdot \left(\frac{\hat{r}}{r^2}\right) d\tau = 4\pi$

if V contains origin

Seem Abstract?

Not really! $\vec{E}(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2}$

for charge q @ origin

Special Vector Fields

$\int_{\vec{a}}^{\vec{b}} \vec{A} \cdot d\vec{l}$ depends on path $\vec{a} \to \vec{b}$

But if $\vec{A} = \nabla f$ then

$\int_{\vec{a}}^{\vec{b}} \vec{A} \cdot d\vec{l} = f(\vec{b}) - f(\vec{a})$

regardless of path

similarly:

$\int_S \vec{F} \cdot d\vec{a}$ depends on surface S

But if $\vec{F} = \nabla \times \vec{A}$

$\int_S \vec{F} \cdot d\vec{a} = \oint \vec{A} \cdot d\vec{l}$

independent of surface

Curl-less / Irrotational Fields

$$\nabla \times \vec{A} = 0 \iff \vec{A} = \pm \nabla f$$

$\Rightarrow \int_{\vec{a}}^{\vec{b}} \vec{A} \cdot d\vec{l}$ independent of path

$\oint \vec{A} \cdot d\vec{l} = 0$ for closed loop

Divergence-less / Solenoidal Fields

$$\nabla \cdot \vec{F} = 0 \iff \vec{F} = \pm \nabla \times \vec{A}$$

$\Rightarrow \int_S \vec{F} \cdot d\vec{a}$ independent of surface

$\Rightarrow \oint \vec{F} \cdot d\vec{a}$ for closed surface

Any Vector Field

$$\vec{F} = \nabla f + \nabla \times \vec{A}$$

for any $\vec{F}$!

= "Helmholtz Theorem"