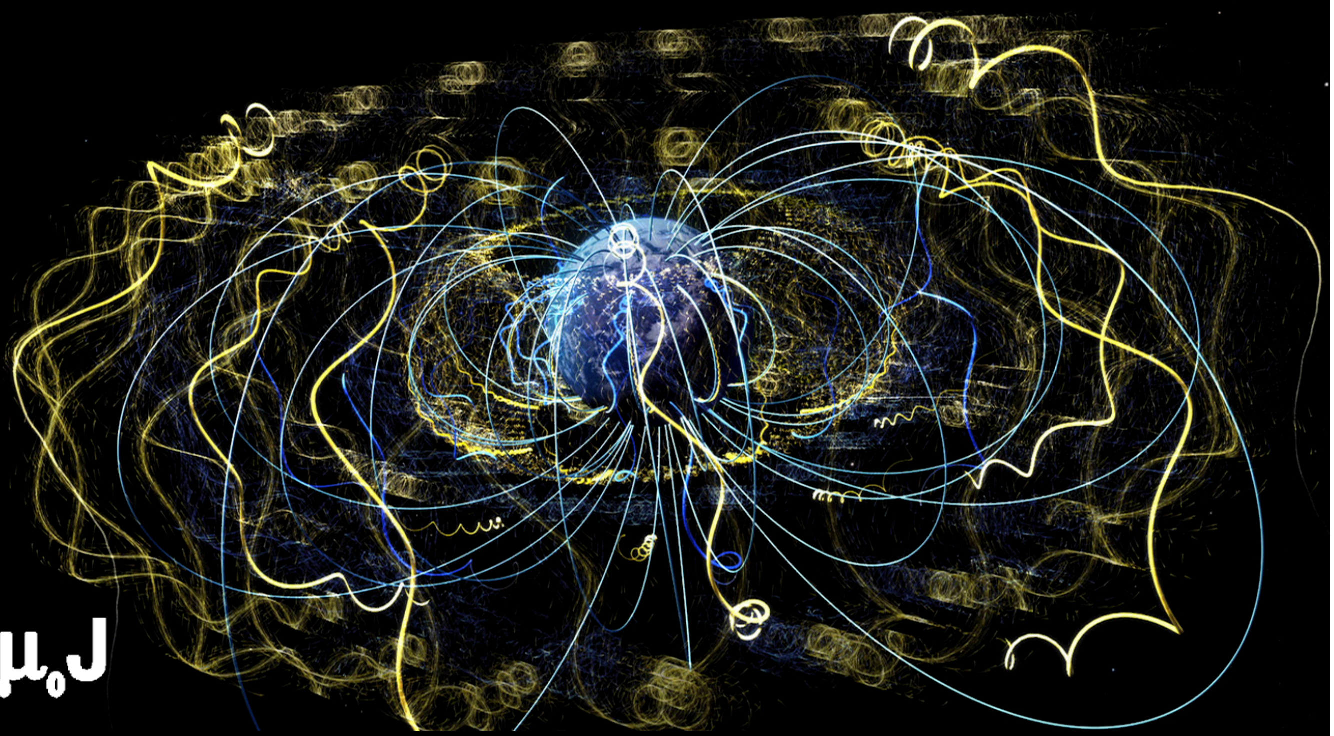
$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

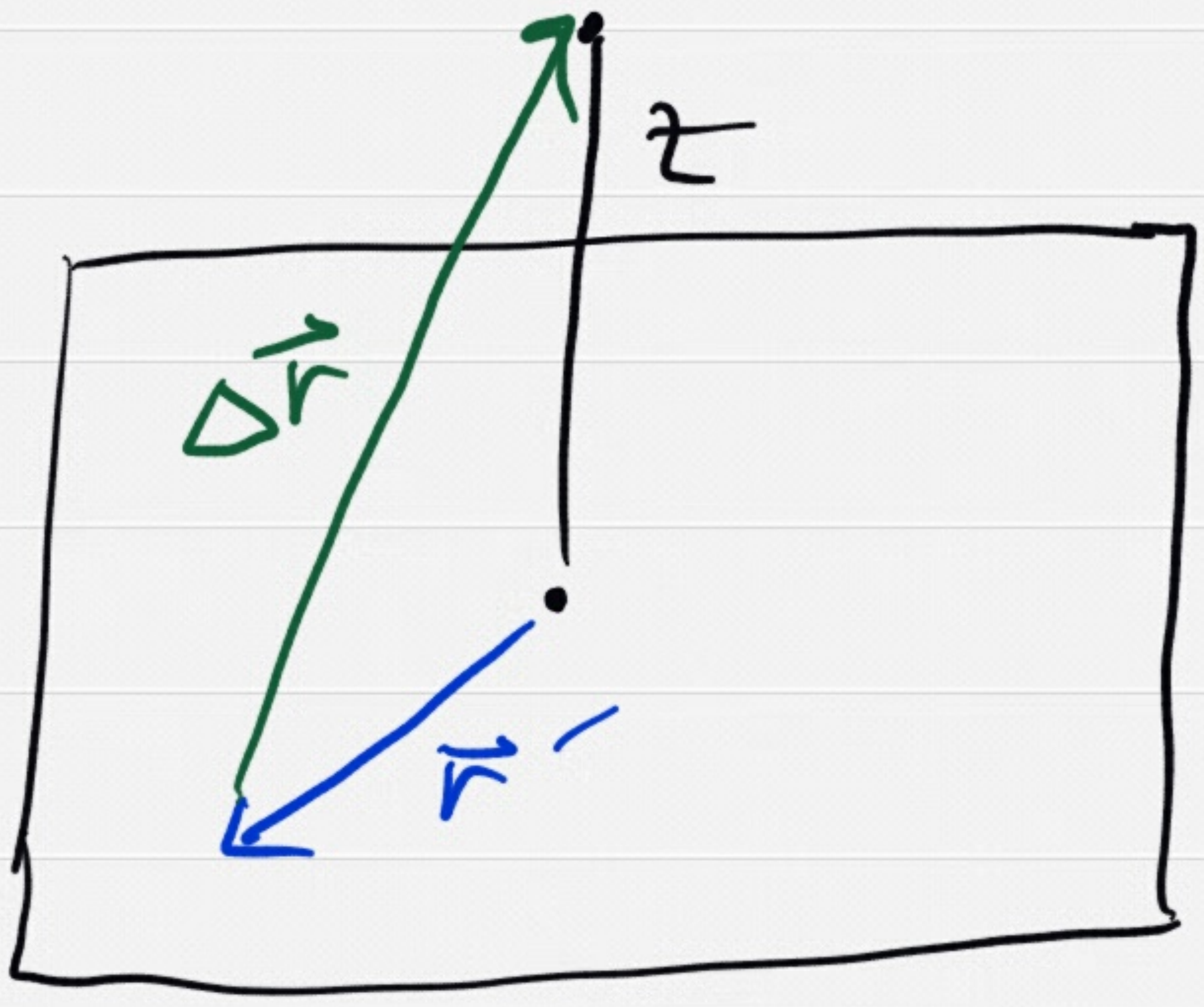
$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}$$



Electricity and Magnetism I: 3811

Professor Jasper Halekas
Van Allen 301
MWF 9:30-10:20 Lecture

Example: Infinite Plane



Charge Density

$\sigma = Q/A$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma(\vec{r}')}{\Delta r^2}\,\hat{\Delta r}\,da'$$

$$\sigma(\vec{r}') = \sigma \qquad , \qquad da' = dx'\,dy'$$

$$\Delta\vec{r} = -x'\hat{x} - y'\hat{y} + z\hat{z}$$

$$\Delta r = \sqrt{x'^2 + y'^2 + z^2}$$

$$\vec{E}(z) = \frac{\sigma}{4\pi\varepsilon_0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{-x'\hat{x} - y'\hat{y} + z\hat{z}}{(x'^2 + y'^2 + z^2)^{3/2}}\,dx'\,dy'$$

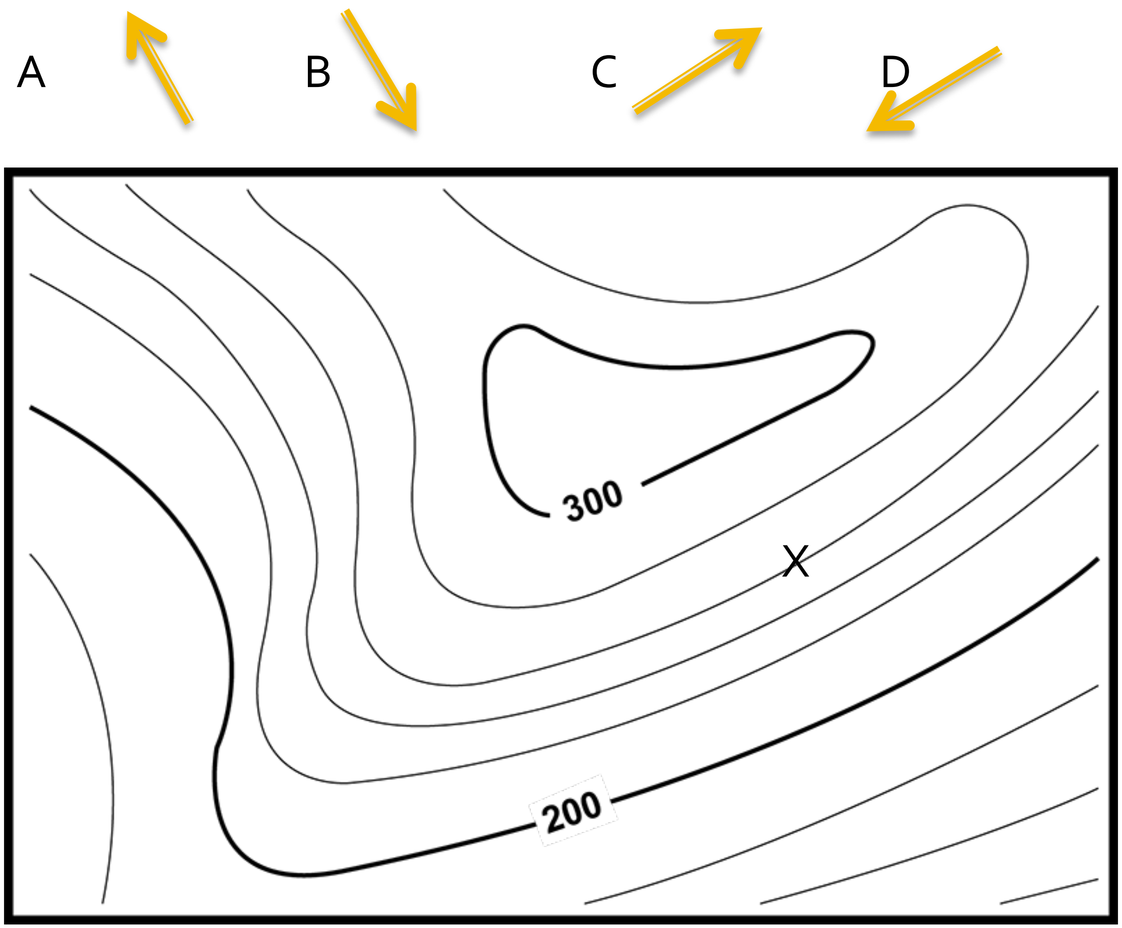
$$\int_{-\infty}^{\infty}\frac{x'\,dx'}{\Delta r^3} = \int\frac{y'\,dy'}{\Delta r^3} = 0 \quad \text{by symmetry}$$

$$\Rightarrow \vec{E}(z) = \frac{\sigma z\hat{z}}{4\pi\varepsilon_0}\underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dx'\,dy'}{(x'^2 + y'^2 + z^2)^{3/2}}}_{= 2\pi/z \text{ after tortuous integration}}$$

$$\Rightarrow \boxed{\vec{E}(z) = \sigma/2\varepsilon_0\;\hat{z}}$$

Check Your Understanding #1

Which direction is the gradient of the scalar function whose contours are shown below, at the point X?



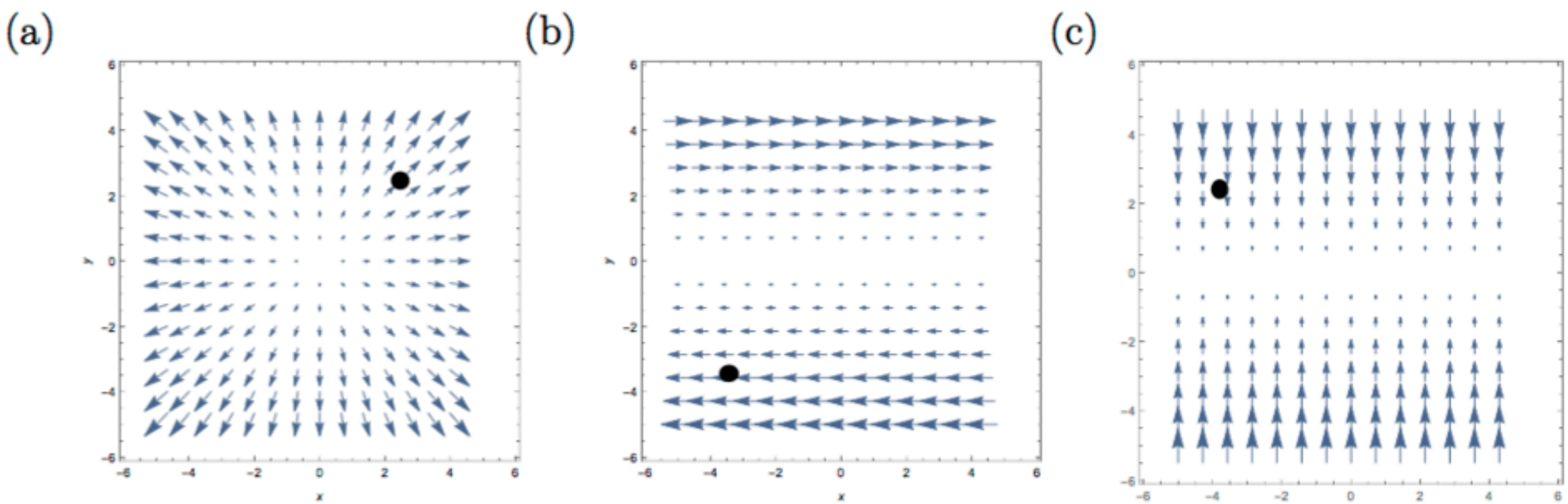
Q1:

Gradient points uphill, perpendicular to contours of constant f

↖ A

Check Your Understanding #2

1. For each of the following vector fields, determine if the divergence is positive, zero, or negative at the indicated point. Then, determine if the curl is out of the board, zero, or into the board.



Q2:

a. $\vec{A} = x\hat{x} + y\hat{y}$

$\nabla \cdot \vec{A} > 0$

$\nabla \times \vec{A} = 0$

b. $\vec{A} = y\hat{x}$

$\nabla \cdot \vec{A} = 0$

$\nabla \times \vec{A}$ into board

c. $\vec{A} = -y\hat{y}$

$\nabla \cdot \vec{A} < 0$

$\nabla \times \vec{A} = 0$

Check Your Understanding #3

- Calculate the line integral of the function $x\hat{y}$ along the straight line from [0,0,0] to [1,1,1].

Q3:

$$\int \vec{A} \cdot d\vec{l}$$

$$= \int A_x\,dx + A_y\,dy + A_z\,dz$$

$$= \int A_y\,dy$$

$$= \int x\,dy$$

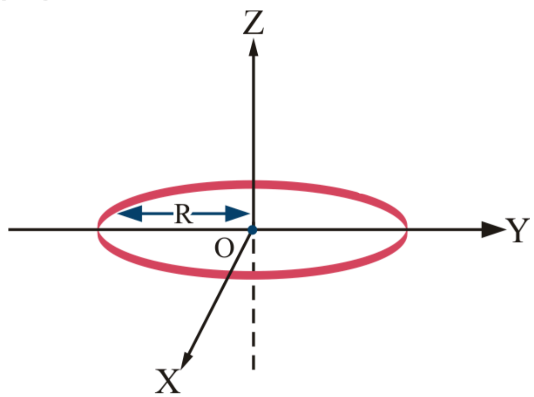
put $x = t$, $y = t$ $\quad t: 0 \to 1$

$dx = dt$, $dy = dt$

$$\Rightarrow \int_0^1 t\,dt = t^2/2 \Big|_0^1 = 1/2$$

Check Your Understanding #4

- Consider a circle of radius R in the x-y plane, centered at the origin. What are the line integrals of the following vector functions taken in a counter-clockwise direction around this ring?
  - The unit vector in the r-direction $\hat{r}$
  - The unit vector in the theta-direction $\hat{\theta}$
  - The unit vector in the phi-direction $\hat{\varphi}$



Q4: $\int \vec{A} \cdot d\vec{l} =$

$\int \vec{A} \cdot R\, d\varphi\, \hat{\varphi}$

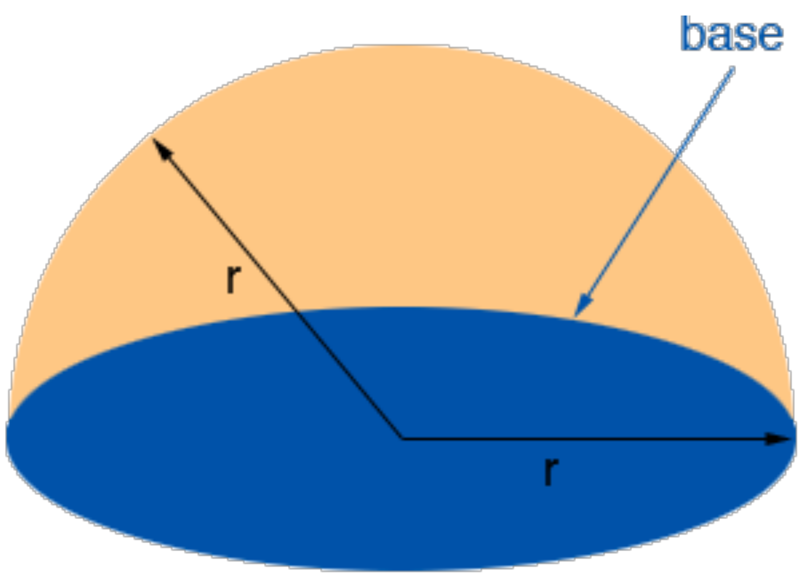
$\Rightarrow \int \hat{r} \cdot d\vec{l} = \boxed{0}$

$\int \hat{\theta} \cdot d\vec{l} = \boxed{0}$

$\int \hat{\varphi} \cdot d\vec{l} = \int_0^{2\pi} R\, d\varphi = \boxed{2\pi R}$

Check Your Understanding #5

- Calculate the surface integral of the function $\hat{z}$ (the unit vector in the z-direction) over a hemisphere of radius r (orange surface shown in image).



Q5: $\nabla \cdot \vec{A} = 0$

$$\Rightarrow \int_{top} \vec{A} \cdot d\vec{a} = \int_{bottom} \vec{A} \cdot d\vec{a}$$

$$= \int \hat{z} \cdot \hat{z} \, da = \int da$$

$$= \boxed{\pi r^2}$$

Or: $\int_{top} \vec{A} \cdot d\vec{a}$

$$= \int \hat{z} \cdot \hat{r} \, da$$

$$= \int \hat{z} \cdot \hat{r} \cdot r^2 \sin\theta \, d\theta \, d\varphi$$

$$= \int r^2 \sin\theta \cos\theta \, d\theta \, d\varphi$$

$$= r^2 \int_0^{2\pi} \left[\int_0^{\pi/2} \sin\theta \cos\theta \right] d\varphi$$

$$= r^2 \int_0^{2\pi} \left[\sin^2\theta/2 \Big|_0^{\pi/2} \right] d\varphi$$

$$= r^2 \int_0^{2\pi} 1/2 \, d\varphi$$

$$= \boxed{\pi r^2}$$

Check Your Understanding #6

- Write the charge density $\rho(\mathbf{r})$ corresponding to a point charge q at the vector position $\mathbf{r}'$, utilizing the three-dimensional Dirac delta function

Q6: $\rho(\vec{r}) = q\,\delta^3(\vec{r}-\vec{r}\,')$