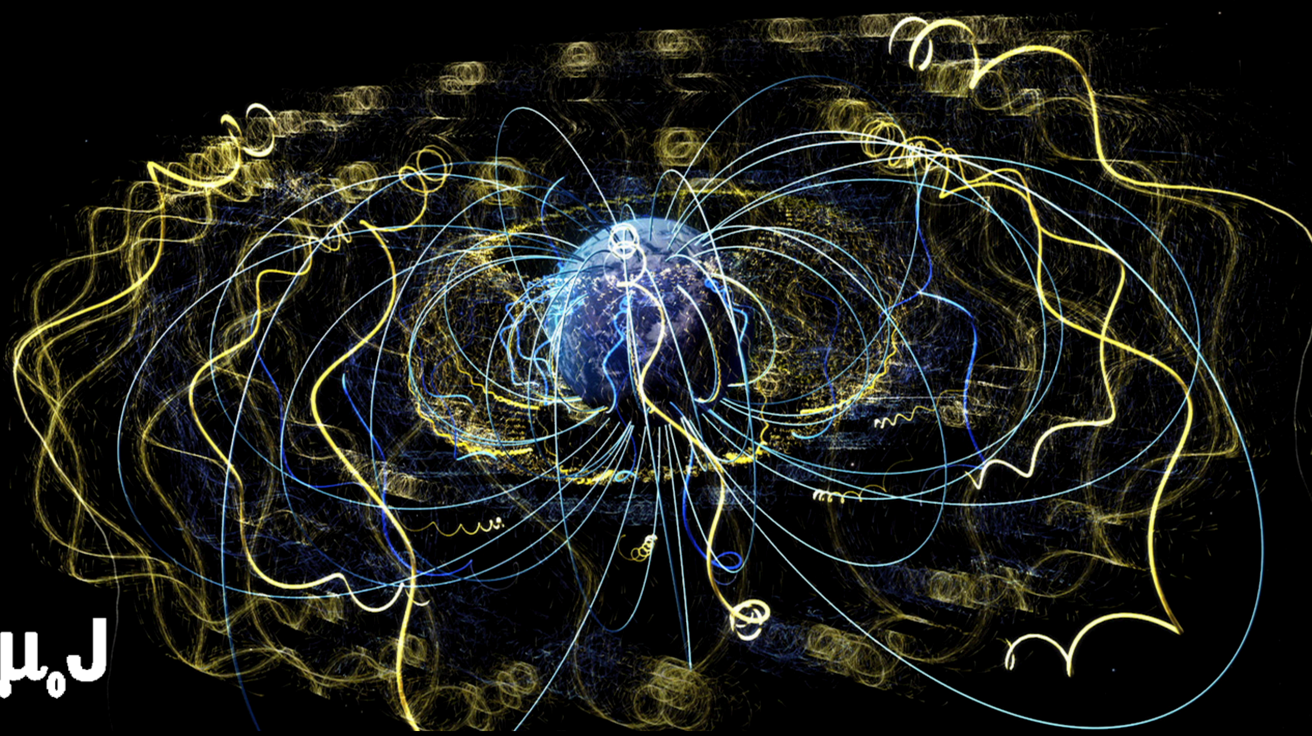
$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}$$



Electricity and Magnetism II: 3812

Professor Jasper Halekas
Van Allen 70
MWF 9:30-10:20 Lecture

Announcements

- No office hours today (2/12)
- Extended office hours tomorrow (2/13)
  - 1:00-3:00pm

$\boxed{8.2 \mid \text{Momentum}}$

Force per unit volume

$$\frac{\vec{F}}{\text{vol}} = \frac{q\vec{E} + q\vec{v}\times\vec{B}}{\text{vol}}$$

$$\Rightarrow \vec{f} = \rho\vec{E} + \vec{J}\times\vec{B}$$

Messy vector math

$$\Rightarrow \vec{f} = \varepsilon_0\left[(\nabla\cdot\vec{E})\vec{E} + (\vec{E}\cdot\nabla)\vec{E}\right] + \frac{1}{\mu_0}\left[(\nabla\cdot\vec{B})\vec{B} + (\vec{B}\cdot\nabla)\vec{B}\right]$$

$$- \nabla\left(\tfrac{1}{2}\varepsilon_0 E^2 + B^2/2\mu_0\right)$$

$$-\mu_0\varepsilon_0\, \partial/\partial t\left((\vec{E}\times\vec{B})/\mu_0\right)$$

"Simplify" using Maxwell Stress Tensor

$$\boxed{T_{ij} = \varepsilon_0\left(E_iE_j - \tfrac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_iB_j - \tfrac{1}{2}\delta_{ij}B^2\right)}$$

$$E_iE_j = \begin{pmatrix} E_x^2 & E_xE_y & E_xE_z \\ E_yE_x & E_y^2 & E_yE_z \\ E_zE_x & E_zE_y & E_z^2 \end{pmatrix}$$

$$\delta_{ij}E^2 = \begin{pmatrix} E^2 & 0 & 0 \\ 0 & E^2 & 0 \\ 0 & 0 & E^2 \end{pmatrix} \quad \text{etc.}$$

$$\nabla\cdot\overleftrightarrow{T} = \begin{pmatrix} \partial/\partial x\, T_{xx} + \partial/\partial y\, T_{yx} + \partial/\partial z\, T_{zx} \\ \partial/\partial x\, T_{xy} + \partial/\partial y\, T_{yy} + \partial/\partial z\, T_{zy} \\ \partial/\partial x\, T_{xz} + \partial/\partial y\, T_{yz} + \partial/\partial z\, T_{zz} \end{pmatrix}$$

$$(\nabla\cdot\overleftrightarrow{T})_j = \sum_i \partial/\partial_i\, T_{ij}$$

$$= \sum_i \partial/\partial_i \Big[\varepsilon_0 (E_i E_j - \tfrac{1}{2}\delta_{ij} E^2) + \tfrac{1}{\mu_0}(B_i B_j - \tfrac{1}{2}\delta_{ij} B^2)\Big]$$

$$= \sum_i \varepsilon_0 \Big[\partial E_i/\partial_i\, E_j + E_i\, \partial E_j/\partial_i - \tfrac{1}{2}\delta_{ij}\, \partial/\partial_i E^2 \Big] + \tfrac{1}{\mu_0}\Big[\partial B_i/\partial_i\, B_j + B_i\, \partial B_j/\partial_i - \tfrac{1}{2}\delta_{ij}\, \partial/\partial_i B^2 \Big]$$

$$= \varepsilon_0 \Big[(\nabla\cdot\vec{E})E_j + (\vec{E}\cdot\nabla)E_j - \tfrac{1}{2}\partial/\partial_j E^2 \Big] + \tfrac{1}{\mu_0}\Big[(\nabla\cdot\vec{B})B_j + (\vec{B}\cdot\nabla)B_j - \tfrac{1}{2}\partial/\partial_j B^2 \Big]$$

$$\Rightarrow \nabla\cdot\overleftrightarrow{T} = \varepsilon_0 \Big[(\nabla\cdot\vec{E})\vec{E} + (\vec{E}\cdot\nabla)\vec{E} - \tfrac{1}{2}\nabla E^2 \Big] + \tfrac{1}{\mu_0}\Big[(\nabla\cdot\vec{B})\vec{B} + (\vec{B}\cdot\nabla)\vec{B} - \tfrac{1}{2}\nabla B^2 \Big]$$

Comparing w/ $\vec{f}$

we find

$$\boxed{\vec{f} = \nabla\cdot\overleftrightarrow{T} - \varepsilon_0\mu_0\, \partial\vec{S}/\partial t}$$

or $$\boxed{\vec{f} = \nabla\cdot\overleftrightarrow{T} - \frac{1}{c^2}\, \partial\vec{S}/\partial t}$$

Total EM Force

$$\vec{F} = \int \vec{f}\, d\tau$$

$$= \int (\nabla \cdot \overleftrightarrow{T})\, d\tau - \epsilon_0 \mu_0 \int \frac{\partial \vec{S}}{\partial t}\, d\tau$$

$$\boxed{\vec{F} = \oint \overleftrightarrow{T} \cdot d\vec{a} - \epsilon_0 \mu_0 \frac{d}{dt} \int \vec{S}\, d\tau}$$

$\vec{F} = d\vec{p}_{mechanical}/dt$ = rate of change in momentum of particles in volume

$\oint \overleftrightarrow{T} \cdot d\vec{a}$ = EM momentum passing through surface (inward)

$\mu_0 \epsilon_0 \int \vec{S}\, d\tau$ = Momentum stored in EM fields in volume

$$\boxed{\begin{aligned} \vec{p}_{EM} &= \mu_0 \epsilon_0 \int \vec{S}\, d\tau \\ &= \frac{1}{c^2} \int \vec{S}\, d\tau \end{aligned}}$$

$$\boxed{\vec{g}_{EM} = \mu_0 \epsilon_0 \vec{S} = \vec{S}/c^2}$$

= EM momentum density

Why is $\vec{g} = \vec{S}/c^2$?

- From relativity

$$E^2 = (pc)^2 + (mc^2)^2$$

- For massless quantity

$$E = pc \quad \text{(i.e. like photon)}$$

$$S = \frac{E}{A \cdot t} = \frac{E}{\text{Area} \cdot \text{Length}} \cdot \frac{\text{Length}}{\text{time}}$$

$$= \frac{E}{\text{volume}} \cdot c$$

$$g = p/\text{volume}$$

$$= \frac{E/c}{\text{volume}}$$

$$= S/c^2 \quad //$$

Continuity of Momentum

If $\vec{p}_M = \text{const.}$
then EM momentum conserved

$$0 = -d\vec{p}_{EM}/dt + \oint \overleftrightarrow{T} \cdot d\vec{a}$$

or $$d\vec{p}_{EM}/dt = \oint \overleftrightarrow{T} \cdot d\vec{a}$$

$\Rightarrow$ Microscopic

$$d/dt \int \vec{g}_{EM}\, d\tau = \oint \overleftrightarrow{T} \cdot d\vec{a}$$

$$\int \partial \vec{g}_{EM}/\partial t \, d\tau = \int (\nabla \cdot \overleftrightarrow{T})\, d\tau$$

True for all volumes

$$\Rightarrow \boxed{\partial \vec{g}_{EM}/\partial t = \nabla \cdot \overleftrightarrow{T}}$$

Continuity Eq. for EM momentum

Note: $-\overleftrightarrow{T}$ is EM momentum flux

Negative since $\oint \overleftrightarrow{T} \cdot d\vec{a}$ is momentum into surface (related to force <u>on</u> surface)

EM Energy and Momentum

$$\boxed{\frac{dW}{dt} = -\frac{d}{dt}\int_{\mathcal{V}} u_{\mathrm{em}}\, d\tau - \oint_{\mathcal{S}} \mathbf{S}\cdot d\mathbf{a}} \quad \longleftarrow\text{-----}\longrightarrow \quad \boxed{\frac{d\mathbf{p}_{\mathrm{mech}}}{dt} = -\epsilon_0\mu_0\frac{d}{dt}\int_{\mathcal{V}} \mathbf{S}\, d\tau + \oint_{\mathcal{S}} \overleftrightarrow{\mathbf{T}}\cdot d\mathbf{a}}$$

$$\frac{\partial}{\partial t}\left(u_{\mathbf{mech}} + u_{\mathbf{em}}\right) = -\nabla\cdot\mathbf{S} \quad \longleftarrow\text{-----}\longrightarrow \quad \frac{\partial}{\partial t}\left(\mathbf{P}_{\mathbf{mech}} + \mathbf{P}_{\mathbf{em}}\right) = -\nabla\cdot\left(-\overleftrightarrow{\mathbf{T}}\right)$$

$$\mathbf{P}_{\mathbf{em}} = \int_{\mathcal{V}}\left(\varepsilon_0\mu_0\mathbf{S}\right)d\tau = \int_{\mathcal{V}}\varepsilon_0\left(\mathbf{E}\times\mathbf{B}\right)d\tau$$

$$u_{\mathrm{em}} = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) \quad \longleftarrow\text{-----}\longrightarrow \quad \mathbf{g} = \varepsilon_0\mu_0\mathbf{S} = \varepsilon_0\left(\mathbf{E}\times\mathbf{B}\right)$$

Poynting Vector S

S : *Energy per unit area (Energy flux density), per unit time transport by EM fields*

$\varepsilon_0\mu_0 S$: *Momentum per unit volume (Momentum density) stored in EM fields*

Stress Tensor $\overleftrightarrow{\mathrm{T}}$

$\overleftrightarrow{\mathrm{T}}$: *EM field stress (Force per unit area) acting on a surface*

$-\overleftrightarrow{\mathrm{T}}$: *Flow of momentum (momentum per unit area, unit time) carried by EM fields*

Continuity Equations of EM fields in empty space

$$\frac{\partial\rho}{\partial t} = -\left(\nabla\cdot\mathbf{J}\right)$$

$$\frac{\partial u_{em}}{\partial t} = -\left(\nabla\cdot\mathbf{S}\right)$$

$(\mathbf{S})$ *playing the part of **J** → **Local conservation of field energy***

$$\frac{\partial\mathbf{g}}{\partial t} = -\nabla\cdot\left(-\overleftrightarrow{\mathbf{T}}\right)$$

$\left(-\overleftrightarrow{\mathbf{T}}\right)$ *playing the part of **J** → **Local conservation of field momentum***

$\boxed{\text{Force}}$

$$\vec{F} = \oint \overleftrightarrow{T} \cdot d\vec{a} - \mu_0 \varepsilon_0 \, d/dt \int \vec{S} \, d\tau$$

$$= \oint \overleftrightarrow{T} \cdot d\vec{a} - d\vec{p}_{EM}/dt$$

In static case

$$\boxed{\vec{F} = \oint \overleftrightarrow{T} \cdot d\vec{a}}$$

T_{ij} = force per unit area in i direction acting on surface w/ normal in j direction

- Terms w/ $i = j$ represent pressure
- Terms w/ $i \neq j$ represent shear

E.g.

$$F_x = \oint (T_{xx} \, da_x + T_{xy} \, da_y + T_{xz} \, da_z)$$

pressure on x surface

x-ward shear on y, z surfaces