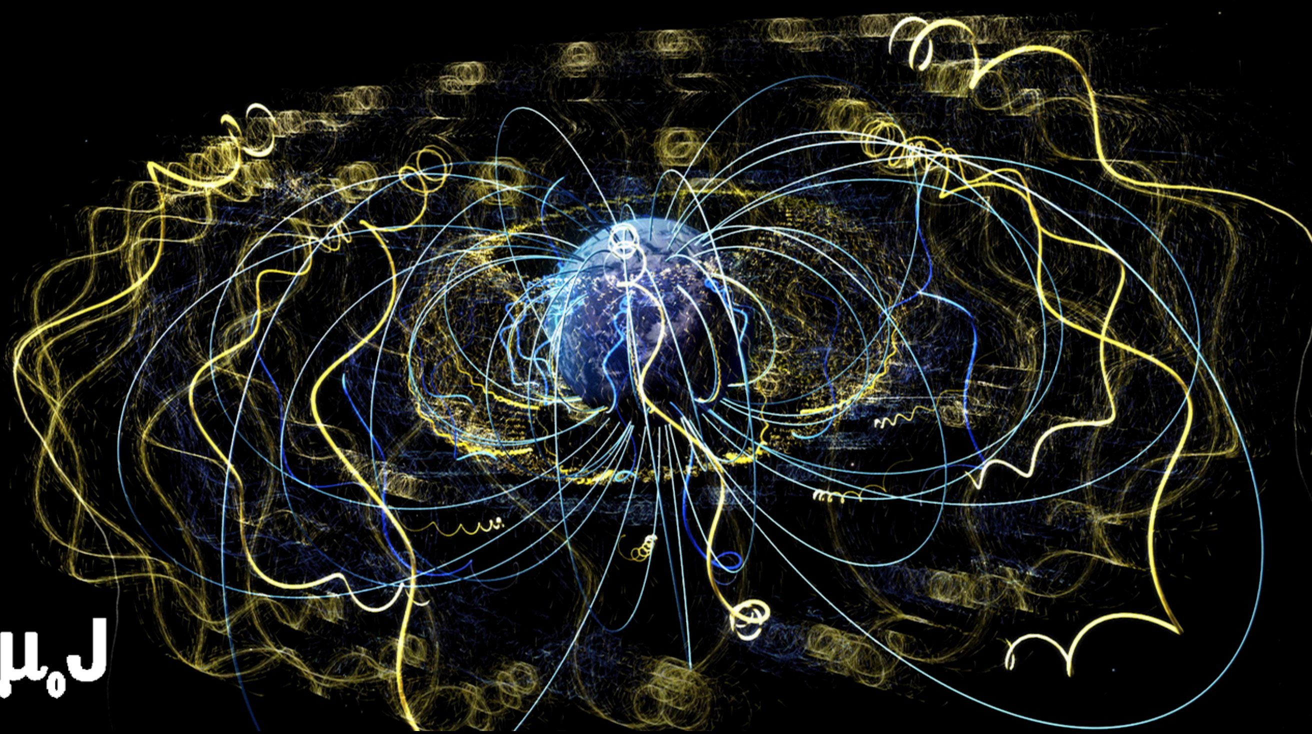
$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

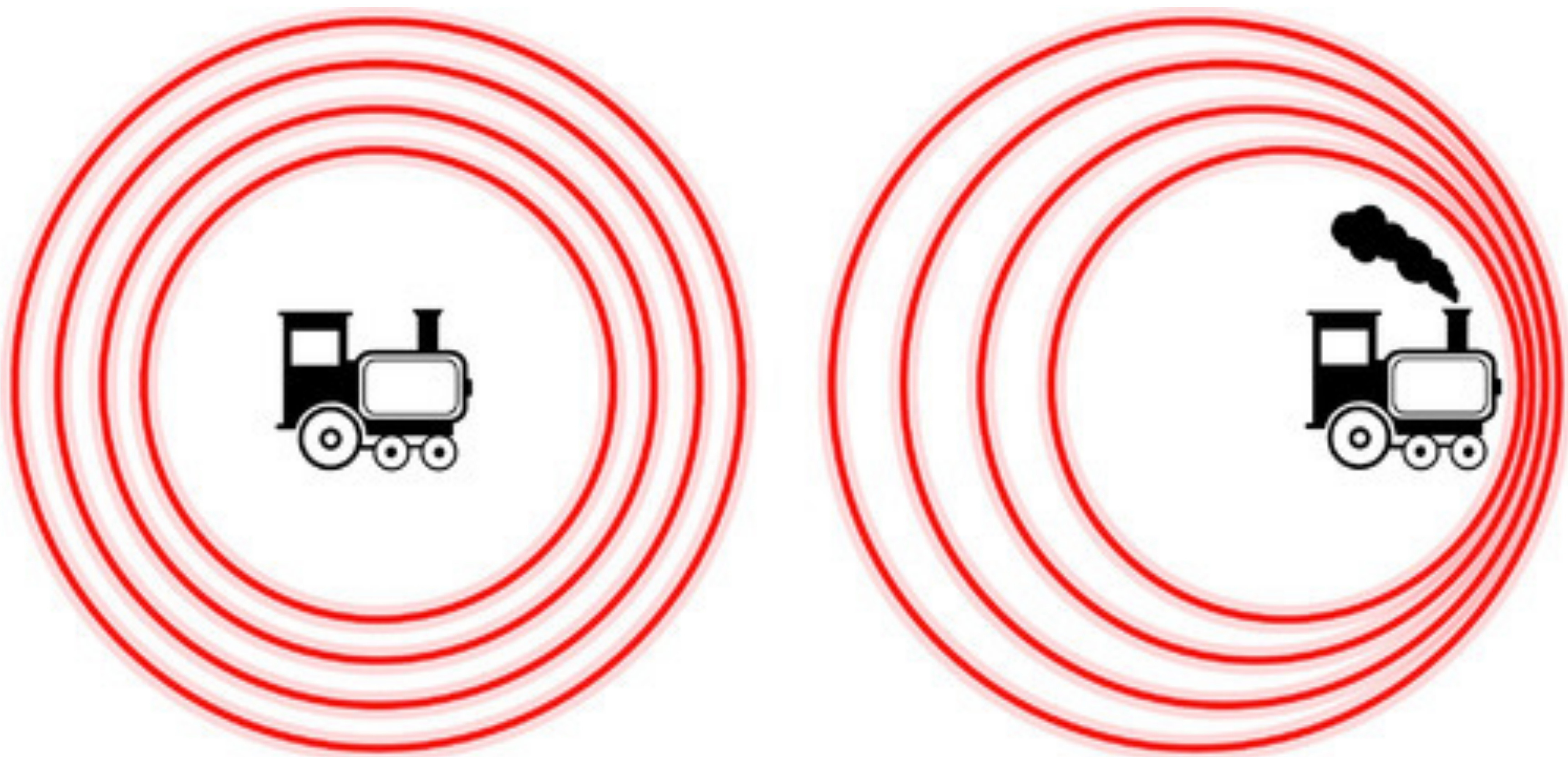
$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}$$



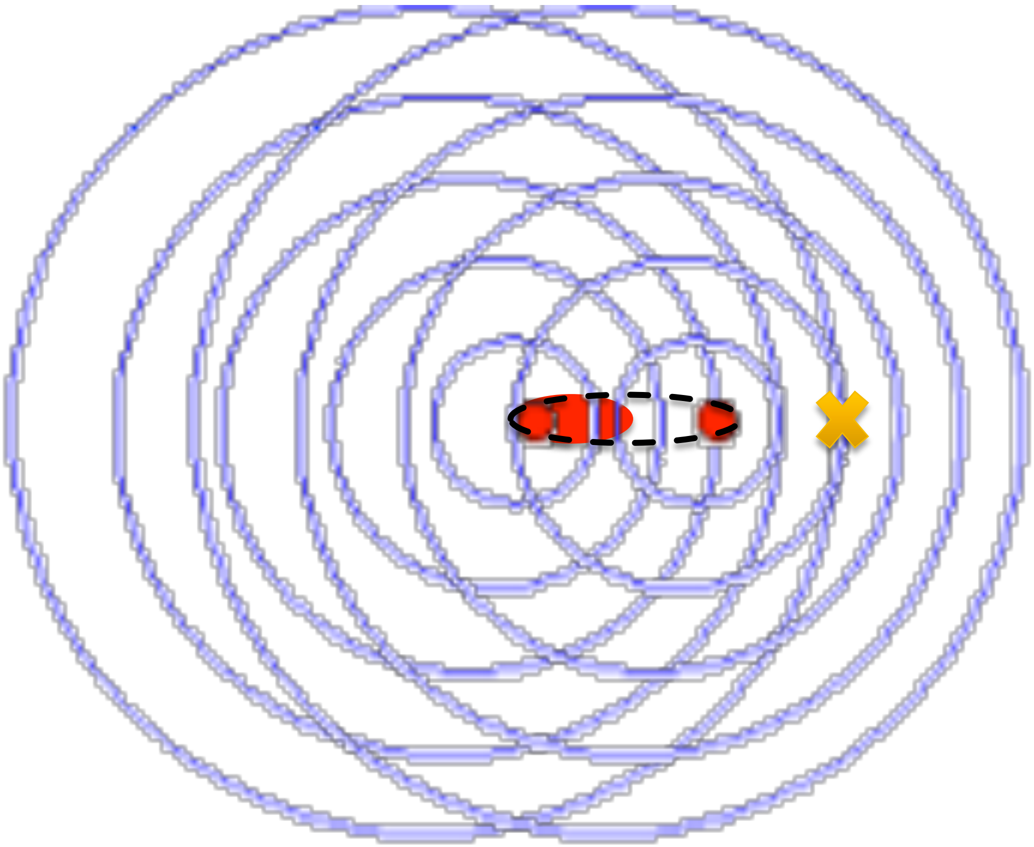
Electricity and Magnetism II: 3812

Professor Jasper Halekas
Van Allen 70
MWF 9:30-10:20 Lecture

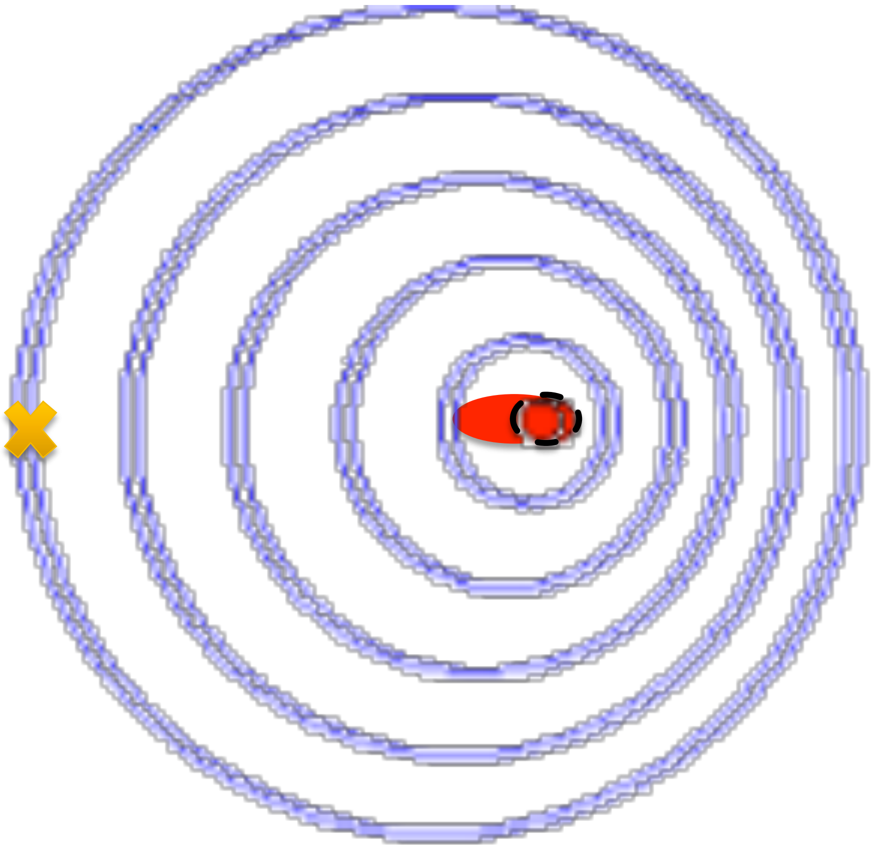
Classical Doppler Shift



Object Moving Toward



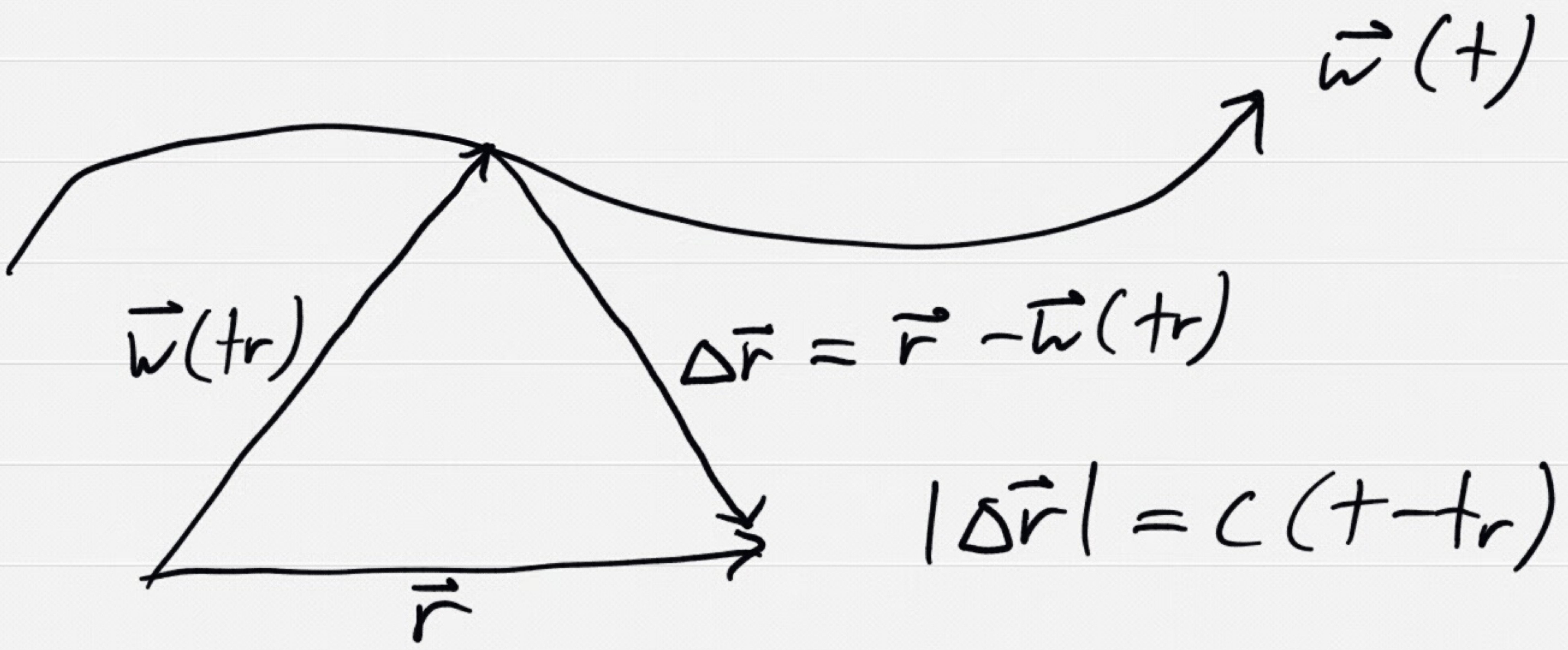
Object Moving Away



10.3 | Point Charges

Liénard-Wiechert Potentials

- Point charge moving along specified trajectory $\vec{w}(t)$
- What are $V(\vec{r},t)$, $\vec{A}(\vec{r},t)$?



- Only one point $\vec{w}(t_r)$ contributes to $V(\vec{r},t)$

$$V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,',t_r)}{\Delta r}\,d\tau'$$

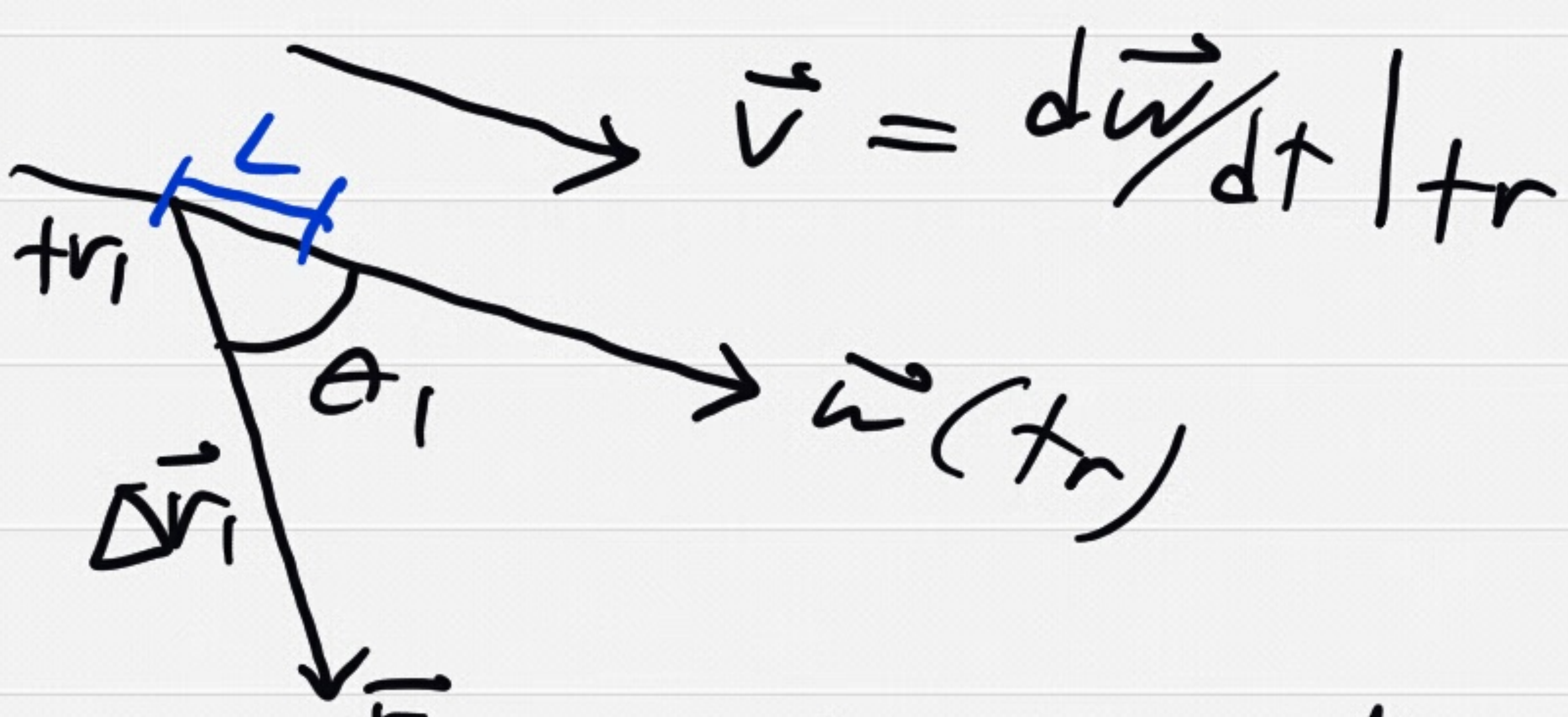
$$= \frac{1}{4\pi\varepsilon_0 \Delta r}\int \rho(\vec{r}\,',t_r)\,d\tau'$$

- You might think

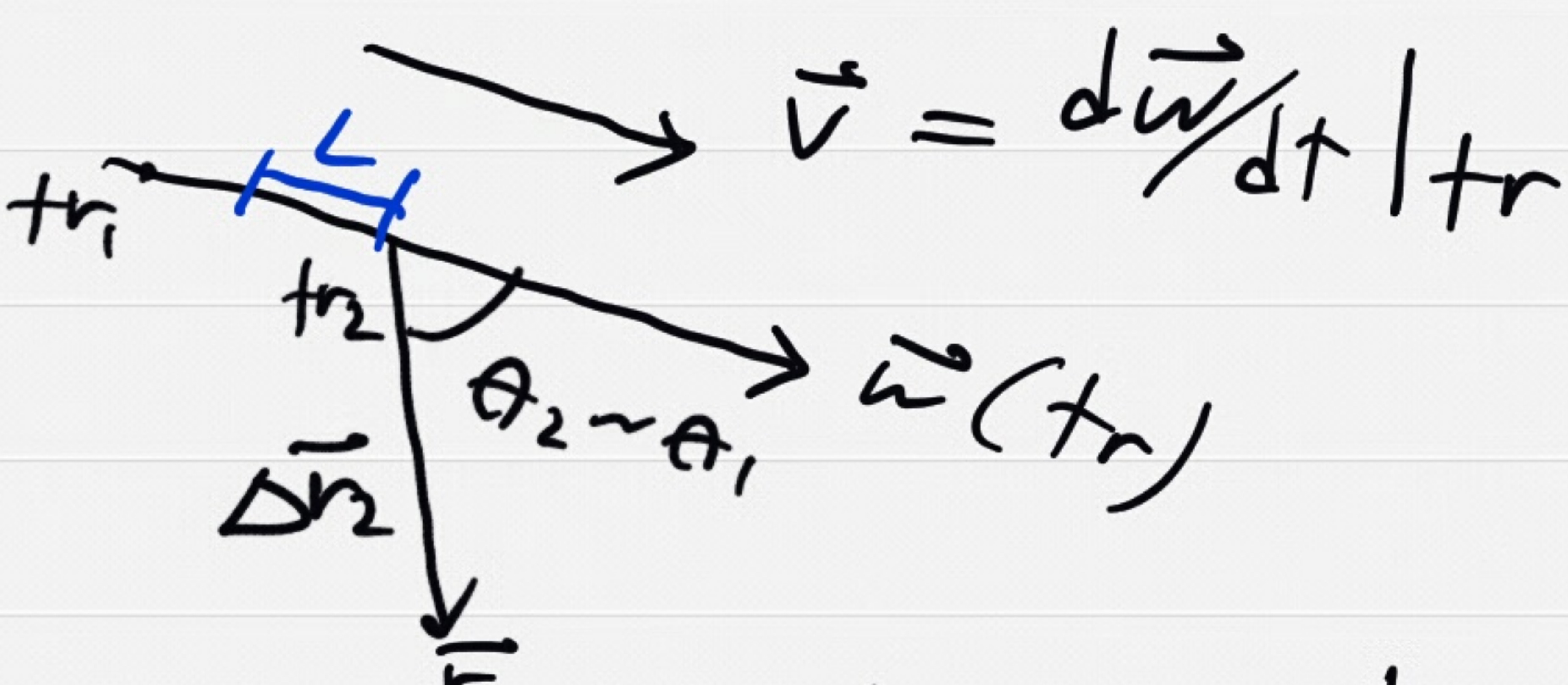
$$\int \rho(\vec{r}', t_r)\, d\tau' = q$$

- But because the charge is moving it is effectively stretched (or compressed) in terms of its effects on an observer

$$\boxed{\text{Volume Stretching}}$$



$$t_{r_1} = t - \Delta r_1 / c$$



$$t_{r_2} = t - \Delta r_2 / c$$

Apparent length $L' = |\vec{\Delta r_2} - \vec{\Delta r_1}|$

Note: This is a non-relativistic effect!

$$L' = L + v\cos\theta\,(t_{r_2} - t_{r_1})$$
$$= L + v\cos\theta\left(\frac{|\Delta\vec{r}_2 - \Delta\vec{r}_1|}{c}\right)$$
$$= L + \frac{v}{c}\cos\theta\, L'$$

$$\Rightarrow L' = L\Big/\left(1 - \frac{v}{c}\cos\theta\right)$$

since $\cos\theta = \Delta\hat{r}\cdot\hat{v}$

$$L' = L\Big/\left(1 - \frac{\Delta\hat{r}\cdot\vec{v}}{c}\right)$$

Transverse dimensions not stretched

so $d\tau' = d\tau\Big/\left(1 - \frac{\Delta\hat{r}\cdot\vec{v}}{c}\right)$

$$\Rightarrow \int \rho(\vec{r}\,', t_r)\, d\tau'$$
$$= q\Big/\left(1 - \frac{\Delta\hat{r}\cdot\vec{v}}{c}\right) = \frac{qc}{c - \Delta\hat{r}\cdot\vec{v}}$$

$$\Rightarrow V(\vec{r}, t) = \frac{1}{4\pi\varepsilon_0 \Delta r}\cdot\frac{qc}{c - \Delta\hat{r}\cdot\vec{v}}$$

or $\boxed{V(\vec{r}, t) = \frac{1}{4\pi\varepsilon_0}\,\frac{qc}{c\,\Delta r - \Delta\vec{r}\cdot\vec{v}}}$

w/ $\vec{v} = d\vec{w}/dt\,\big|_{t_r}$

$$\Delta\vec{r} = \vec{r} - \vec{w}(t_r)$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}, t_r)}{\Delta r}\, d\tau'$$

$$= \frac{\mu_0}{4\pi\Delta r}\int \rho(\vec{r}', t_r)\,\vec{v}(t_r)\, d\tau'$$

$$\boxed{\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\frac{qc\vec{v}}{c\Delta r - \vec{\Delta r}\cdot\vec{v}}}$$

$$\frac{\mu_0}{4\pi} \Big/ \frac{1}{4\pi\varepsilon_0} = \mu_0\varepsilon_0 = 1/c^2$$

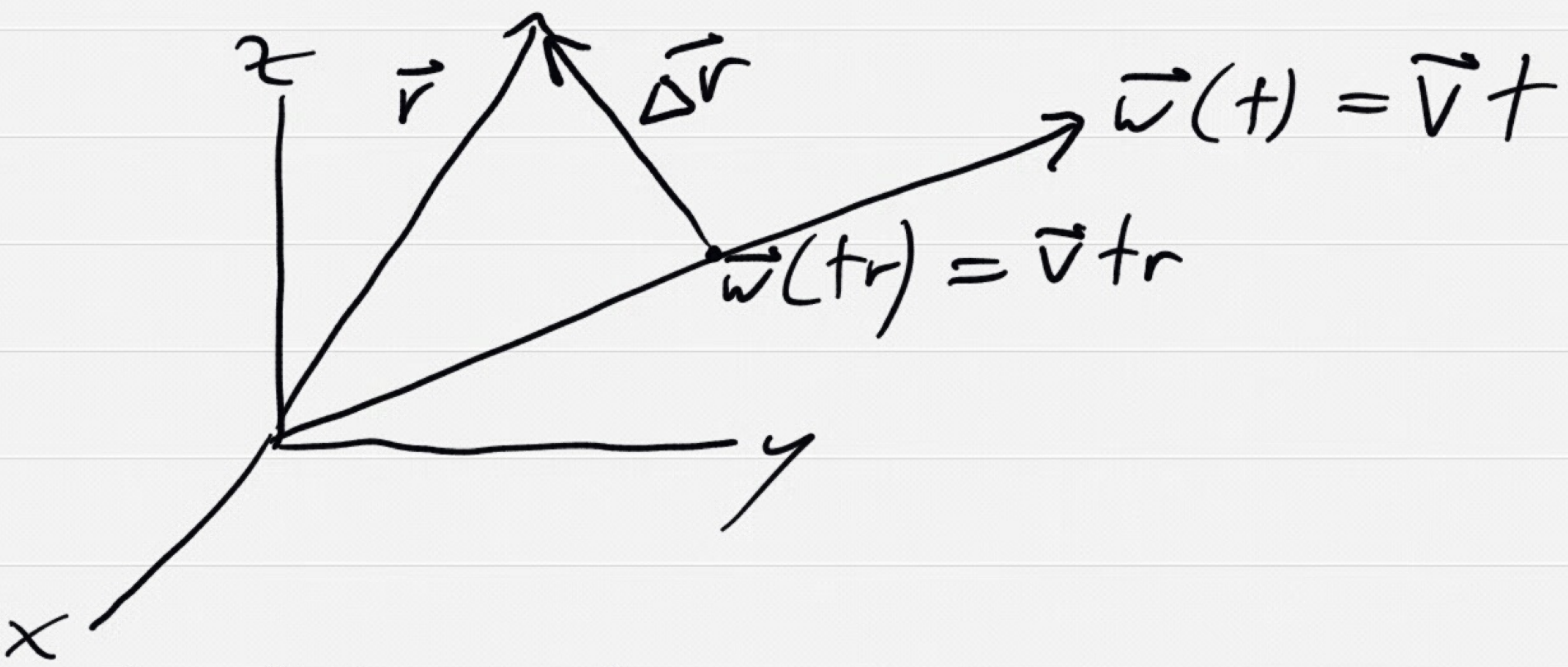
So $\boxed{\vec{A}(\vec{r},t) = \frac{\vec{v}}{c^2} V(\vec{r},t)}$

For a moving point charge

Charge in Uniform Motion

$\vec{w}(t) = \vec{v}t$ w/ $\vec{v}$ constant

say $\vec{w}(0) = [0, 0, 0]$



$$|\vec{\Delta r}| = |\vec{r} - \vec{v}t_r| = c(t - t_r)$$

$$\Rightarrow (\vec{r} - \vec{v}t_r)\cdot(\vec{r} - \vec{v}t_r) = c^2(t - t_r)^2$$

$$\Rightarrow r^2 + v^2 t_r^2 - 2\vec{r}\cdot\vec{v}\,t_r = c^2t^2 + c^2t_r^2 - 2c^2 t\,t_r$$

quadratic formula

$$\Rightarrow t_r = \frac{(c^2 t - \vec{r}\cdot\vec{v}) - \sqrt{(c^2t - \vec{r}\cdot\vec{v})^2 + (c^2 - v^2)(r^2 - c^2t^2)}}{c^2 - v^2}$$

$$\Delta r = |\vec{\Delta r}| = c(t - t_r)$$

$$\hat{\Delta r} = \frac{\vec{\Delta r}}{\Delta r} = \frac{\vec{r} - \vec{v}t_r}{c(t - t_r)}$$

$$\Rightarrow \Delta r\left(1-\frac{\hat{\Delta r}\cdot\vec{v}}{c}\right)$$

$$= c(t-t_r)\left[1-\frac{\vec{v}}{c}\cdot\frac{\vec{r}-\vec{v}t_r}{c(t-t_r)}\right]$$

$$= c(t-t_r)-\frac{\vec{v}\cdot\vec{r}}{c}+\frac{v^2}{c}t_r$$

$$= \frac{1}{c}\left[(c^2t-\vec{r}\cdot\vec{v})-(c^2-v^2)t_r\right]$$

$$= \frac{1}{c}\sqrt{(c^2t-\vec{r}\cdot\vec{v})^2+(c^2-v^2)(r^2-c^2t^2)}$$

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\frac{qc}{c\left(\Delta r\left(1-\frac{\hat{\Delta r}\cdot\vec{v}}{c}\right)\right)}$$

$$\Rightarrow \boxed{V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\frac{qc}{\sqrt{(c^2t-\vec{r}\cdot\vec{v})^2+(c^2-v^2)(r^2-c^2t^2)}}}$$

$$\& \boxed{\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\frac{qc\vec{v}}{\sqrt{(c^2t-\vec{r}\cdot\vec{v})^2+(c^2-v^2)(r^2-c^2t^2)}}}$$

Yikes!

Low-Velocity Limit

$$(c^2 t - \vec{r}\cdot\vec{v})^2 + (c^2 - v^2)(r^2 - c^2 t^2)$$

$$\sim c^4 t^2 - 2c^2 t\,\vec{r}\cdot\vec{v} + c^2 r^2 - c^4 t^2 + v^2 c^2 t^2$$

for $v << c$

$$= c^2 (v^2 t^2 + r^2 - 2t\,\vec{r}\cdot\vec{v})$$

$$= c^2 |\vec{r} - \vec{v}t|^2$$

$$\Rightarrow V(\vec{r}, t) \sim \frac{qc}{4\pi\varepsilon_0} \frac{1}{c|\vec{r} - \vec{v}t|}$$

$$= \frac{q}{4\pi\varepsilon_0 R}$$

w/ R distance from current (not retarded) position of charge