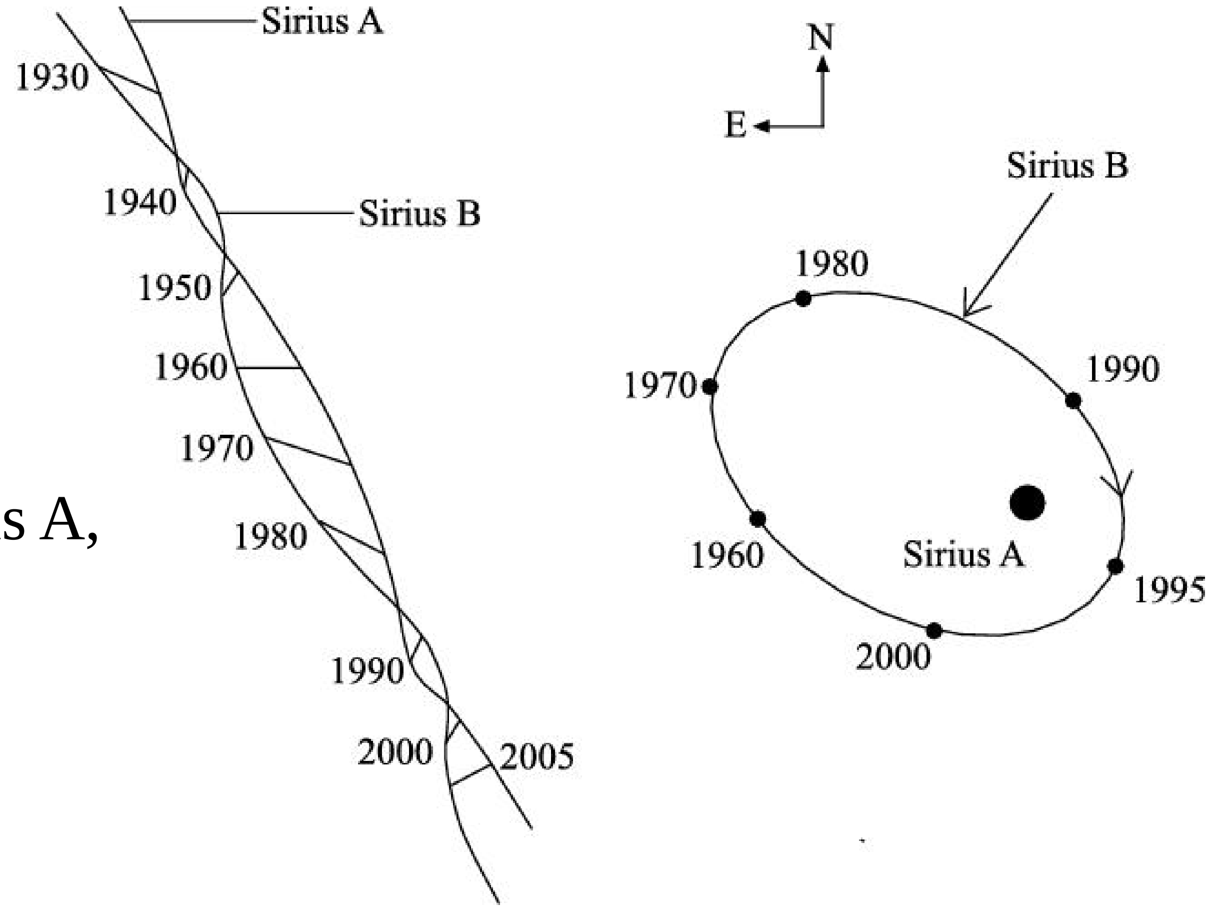
Outline

- Sirius B
- Equation of state of a degenerate gas
- Properties of white dwarfs

Sirius B



- Sirius B is companion to Sirius A, which is the brightest star.
- Bessel in 1844 found Sirius' proper motion implies a companion star.
- Clark in 1915, measured spectrum of Sirius B. From luminosity and temperature calculate radius of ~ 6000 km.
- First white dwarf discovered was actually 40 Eridani B.

How Dense is Quantum?

- Need to use quantum statistics when average separation between particles is comparable to the de Broglie wavelength $\lambda = h/p$.
- Use $E = p^2/2m = (3/2)\,kT$, find $\lambda = h(3mkT)^{-1/2}$.
  - Note λ is larger for electrons, since $m_e << m_p$.
  - Quantum statistics will become important first for electrons.
  - Electrons will “become degenerate” first.
- Corresponding density $\rho = m_p/(\text{volume per particle}) = m_p/(\lambda/2)^3$.
  - Note use m_p for density, but m_e for λ.
- For core of Sun to be degenerate need $\rho > 640$ g cm^{-3}
  - classical treatment is ok since $\rho_C = 150$ g cm^{-3}

Phase Space

- Count states in “phase space” - 6d space of momentum and position.
- Number of states

$$dN = f(p)\frac{d^3\vec{p}\,d^3\vec{x}}{h^3} = f(p)\frac{d^3\vec{p}\,dV}{h^3}$$

- Unit volume is h^3. Can motivate from Heisenburg uncertainty principle or derive from solutions of Schrödinger equation.
  - $\Delta x\, \Delta p > h \rightarrow \Delta x\, \Delta p_x\;\; \Delta y\, \Delta p_y\, \Delta z\, \Delta p_z > h^3 \rightarrow d^3p\; dV > h^3$
- Function $f(p)$ is occupation number = probability that state at given momentum is occupied.
- Divide by volume to get number density of particles
  - $n(p)\; dp = f(p)\; d^3p$

Classical versus quantum statistics

- Classical occuption number function

$$f(p) = \frac{1}{e^{(E-\mu)/kT}}$$

where μ is the "chemical potential".

- This leads to the Maxwell-Boltzmann distribution

$$n(p)\, d^3\vec{p} \quad = \quad 4\pi \left(\frac{m}{2\pi kT} \right)^{3/2} \frac{p^2}{m^3} e^{-p^2/2mkT}\, dp$$

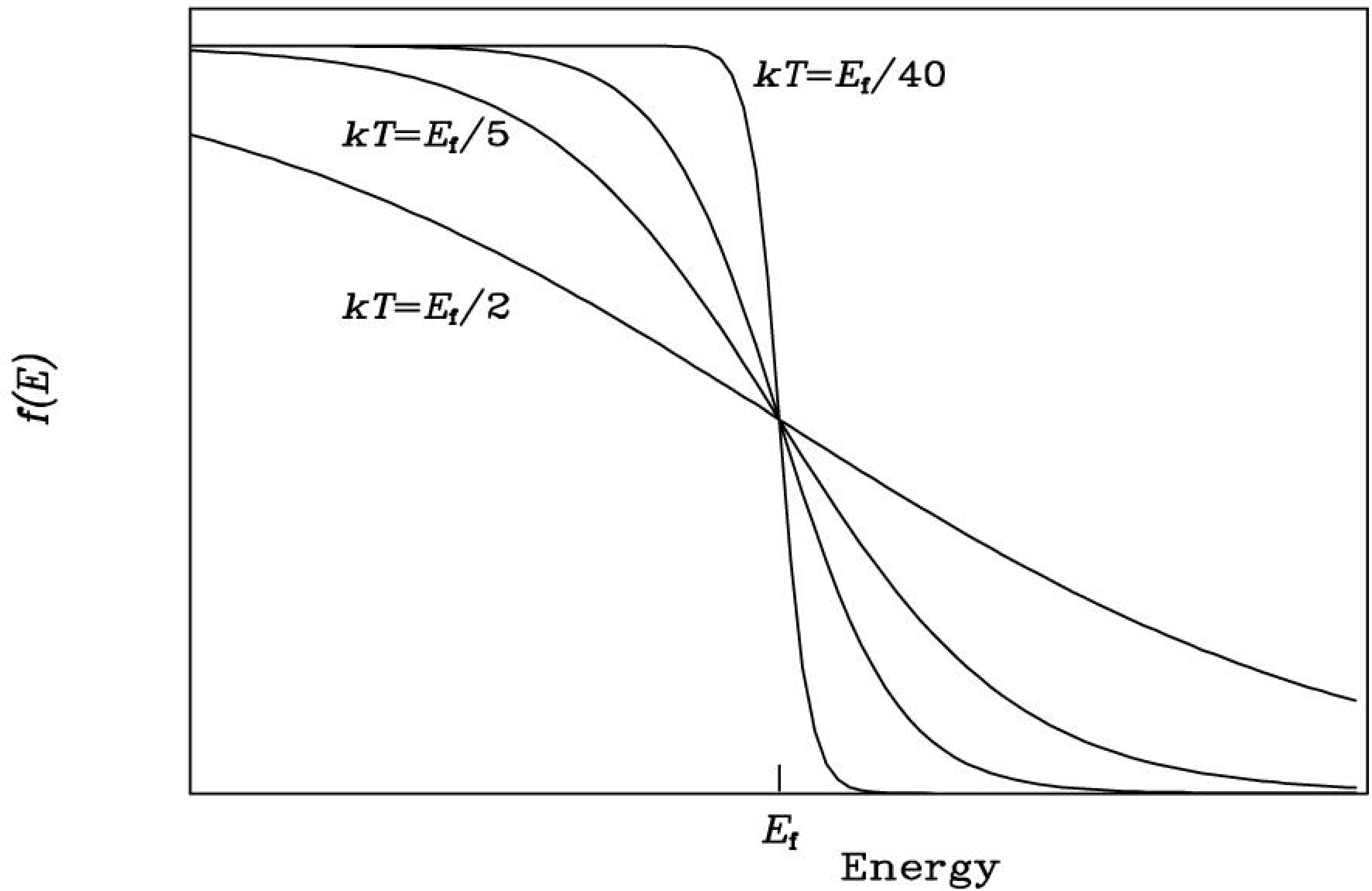
Fermi-Dirac Statistics

- Quantum particles are bosons (integer spin) that follow Bose-Einstein statistics or fermions (half integer spin) that follow Fermi-Dirac statistics.
- Fermions are unneighborly, two identical fermions cannot occupy the same quantum state. This leads to occupation number function

$$f(p) = \frac{1}{e^{(E-\mu)/kT} + 1}$$

- In general, chemical potential is a function of energy. We will consider the case where μ(E) $\rightarrow$ E_f, where E_f is the “Fermi energy”.
- If $kT << E_f$, then f(E) is a step function with states $E < E_f$ occupied and states $E > E_f$ empty.
- Define Fermi momentum p_f so that $p_f^2/2m = E_f$

Fermi-Dirac Statistics



Fermi-Dirac Statistics

- In limit $kT << E_f$, integrals become easy.
- Number density of particles
  - $n(p)\,dp = f(p)\,d^3p = 8\pi p^2 dp/h^3$ for $p \leq p_f$,

    $= 0$ for $p > p_f$

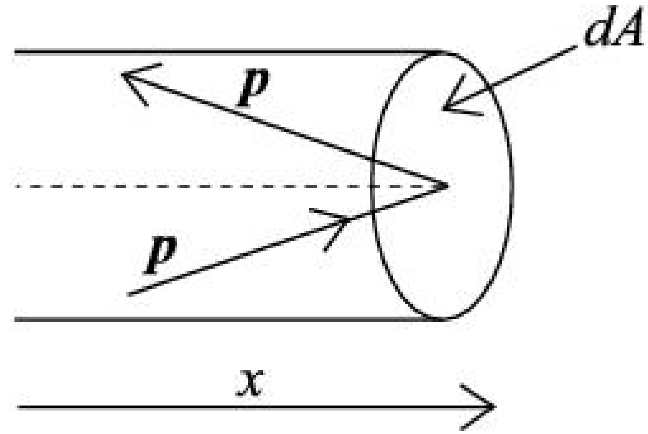
- Find total particle number density by integrating

$$n = \int_0^\infty f(p)\,d^3\vec{p} = \int_0^{p_f} \frac{8\pi}{h^3} p^2 dp = \frac{8\pi}{3h^3} p_f^3$$

Pressure

- To find pressure, note that particle with momentum p transfers 2p to wall during a collision

$$P=\frac{dF_x}{dA}=\frac{2p_x}{dA\,dt}=\frac{2p_x}{dA}\frac{v_x}{dx}=\frac{2p_x v_x}{dV}$$



- Pressure is then (note that half of particles are going the wrong way)

$$P = \int_0^\infty dN(p)\frac{p_x v_x}{dV}dp = \frac{1}{3}\int_0^\infty n(p)\,p\,v\,dp$$

- where we have assumed isotropic velocities and used

$$p_x v_x = mv_x^2 = \frac{1}{3}mv^2 = \frac{1}{3}pv$$

Pressure

- Use $n(p)\,dp = 8\pi p^2 dp/h^3$ for $p \le p_f$, to find calculate pressure

$$P = \frac{1}{3}\int_0^\infty n(p)\, p\, v\, dp = \frac{1}{3}\int_0^{p_f} \frac{8\pi}{h^3}\frac{p^4}{m}\, dp = \frac{8\pi}{3h^3}\frac{p_f^5}{5m}$$

- Recall $n = \dfrac{8\pi}{3h^3} p_f^3$
- Then

$$P = \frac{8\pi}{3h^3 m}\left(\frac{3h^3}{8\pi}\right)^{5/3}\frac{1}{5} n_e^{5/3} = \left(\frac{3}{8\pi}\right)^{2/3}\frac{h^2}{5m} n_e^{5/3}$$

- Note $n_e = Zn_+ = Z\rho/Am_p$, so equation of state for non-relativistic degenerate electron gas is:

$$P = \left(\frac{3}{\pi}\right)^{2/3}\frac{h^2}{20m} m_p^{-5/3}\left(\frac{Z}{A}\right)^{5/3}\rho^{5/3}$$

Pressure

- For typical white dwarf, $\rho \sim 10^6$ g/cm^3, $Z/A \sim 0.5$

$$P = \left(\frac{3}{\pi}\right)^{2/3} \frac{h^2}{20m} m_p^{-5/3} \left(\frac{Z}{A}\right)^{5/3} \rho^{5/3} = 3\times 10^{22} \text{ dyne cm}^{-2}$$

- Compare to thermal pressure of nuclei with $T \sim 10^7$ K,

$$P = n_+ kT = 2\times 10^{20} \text{ dyne cm}^{-2}$$

- Note it is possible to derive the ideal gas law from the Maxwell-Boltzmann distribution using the same equation

$$P = \frac{1}{3}\int_0^\infty n(p)\, p\, v\, dp = nkT$$

Homework

- For next class:
  - Problem 4-1