Outline

• Sirius B
• Equation of state of a degenerate gas
• Properties of white dwarfs
Sirius B

- Sirius B is companion to Sirius A, which is the brightest star.
- Bessel in 1844 found Sirius' proper motion implies a companion star.
- Clark in 1915, measured spectrum of Sirius B. From luminosity and temperature calculate radius of ~6000 km.
- First white dwarf discovered was actually 40 Eridani B.
How Dense is Quantum?

- Need to use quantum statistics when average separation between particles is comparable to the de Broglie wavelength $\lambda = h/p$.

- Use $E = p^2/2m = (3/2) kT$, find $\lambda = h(3mkT)^{-1/2}$.
  - Note $\lambda$ is larger for electrons, since $m_e << m_p$.
  - Quantum statistics will become important first for electrons.
  - Electrons will “become degenerate” first.

- Corresponding density $\rho = m_p/(\text{volume per particle}) = m_p/(\lambda/2)^3$.
  - Note use $m_p$ for density, but $m_e$ for $\lambda$.

- For core of Sun to be degenerate need $\rho > 640 \text{ g cm}^{-3}$
  - classical treatment is ok since $\rho_C = 150 \text{ g cm}^{-3}$
Phase Space

- Count states in “phase space” - 6d space of momentum and position.
- Number of states

\[ dN = f(p) \frac{d^3 \vec{p} \, d^3 \vec{x}}{h^3} = f(p) \frac{d^3 \vec{p} \, dV}{h^3} \]

- Unit volume is \( h^3 \). Can motivate from Heisenburg uncertainty principle or derive from solutions of Schrödinger equation.
  - \( \Delta x \, \Delta p > h \rightarrow \Delta x \, \Delta p_x \, \Delta y \, \Delta p_y \, \Delta z \, \Delta p_z > h^3 \rightarrow d^3 p \, dV > h^3 \)
- Function \( f(p) \) is occupation number = probability that state at given momentum is occupied.
- Divide by volume to get number density of particles
  - \( n(p) \, dp = f(p) \, d^3 p \)
Classical versus quantum statistics

- Classical occupation number function

\[ f(p) = \frac{1}{e^{(E-\mu)/kT}} \]

where \( \mu \) is the "chemical potential".

- This leads to the Maxwell-Boltzmann distribution

\[ n(p) d^3 \vec{p} = 4\pi \left( \frac{m}{2\pi kT} \right)^{3/2} \frac{p^2}{m^3} e^{-p^2/2mkT} dp \]
Fermi-Dirac Statistics

- Quantum particles are bosons (integer spin) that follow Bose-Einstein statistics or fermions (half integer spin) that follow Fermi-Dirac statistics.
- Fermions are unneighborly, two identical fermions cannot occupy the same quantum state. This leads to occupation number function

\[ f(p) = \frac{1}{e^{(E-\mu)/kT} + 1} \]

- In general, chemical potential is a function of energy. We will consider the case where \( \mu(E) \to E_f \), where \( E_f \) is the “Fermi energy”.
- If \( kT \ll E_f \), then \( f(E) \) is a step function with states \( E < E_f \) occupied and states \( E > E_f \) empty.
- Define Fermi momentum \( p_f \) so that \( p_f^2/2m = E_f \).
Fermi-Dirac Statistics

\[ f(E) \]

\[ kT = \frac{E_t}{40} \]

\[ kT = \frac{E_t}{5} \]

\[ kT = \frac{E_t}{2} \]

Energy
Fermi-Dirac Statistics

- In limit $kT \ll E_f$, integrals become easy.

- Number density of particles
  
  - $n(p) \, dp = f(p) \, d^3p = \frac{8\pi p^2 \, dp}{h^3}$ for $p \leq p_f$
  
  - $= 0$ for $p > p_f$

- Find total particle number density by integrating

  $$n = \int_0^\infty f(p) \, d^3\vec{p} = \int_0^{p_f} \frac{8\pi}{h^3} p^2 \, dp = \frac{8\pi}{3h^3} p_f^3$$
Pressure

- To find pressure, note that particle with momentum $p$ transfers $2p$ to wall during a collision
  \[ P = \frac{dF_x}{dA} = \frac{2 p_x}{dA \, dt} = \frac{2 p_x v_x}{dA \, dx} = \frac{2 p_x v_x}{dV} \]

- Pressure is then (note that half of particles are going the wrong way)
  \[ P = \int_0^\infty dN(p) \frac{p_x v_x}{dV} \, dp = \frac{1}{3} \int_0^\infty n(p) p \, v \, dp \]

- where we have assumed isotropic velocities and used
  \[ p_x v_x = mv_x^2 = \frac{1}{3} mv^2 = \frac{1}{3} pv \]
Pressure

- Use \( n(p) \, dp = \frac{8\pi p^2 dp}{h^3} \) for \( p \leq p_f \), to find calculate pressure

\[
P = \frac{1}{3} \int_0^{p_f} n(p) p \, dp = \frac{1}{3} \int_0^{p_f} \frac{8\pi p^4}{h^3} \, dp = \frac{8\pi}{3} \frac{p_f^5}{5m}
\]

- Recall \( n = \frac{8\pi}{3h^3} p_f^3 \)

- Then

\[
P = \frac{8\pi}{3h^3 m} \left( \frac{3h^3}{8\pi} \right)^{5/3} \frac{1}{5} n_e^{5/3} = \left( \frac{3}{8\pi} \right)^{2/3} \frac{h^2}{5m} n_e^{5/3}
\]

- Note \( n_e = Zn_+ = Z \rho/Am_p \), so equation of state for non-relativistic degenerate electron gas is:

\[
P = \left( \frac{3}{\pi} \right)^{2/3} \frac{h^2}{20m} m_p^{-5/3} \left( \frac{Z}{A} \right)^{5/3} \rho^{5/3}
\]
Pressure

- For typical white dwarf, $\rho \sim 10^6 \text{ g/cm}^3$, $Z/A \sim 0.5$

$$P = \left(\frac{3}{\pi}\right)^{2/3} \frac{\hbar^2}{20m} m_p^{-5/3} \left(\frac{Z}{A}\right)^{5/3} \rho^{5/3} = 3 \times 10^{22} \text{ dyne cm}^{-2}$$

- Compare to thermal pressure of nuclei with $T \sim 10^7 \text{ K}$,

$$P = n_+ kT = 2 \times 10^{20} \text{ dyne cm}^{-2}$$

- Note it is possible to derive the ideal gas law from the Maxwell-Boltzmann distribution using the same equation

$$P = \frac{1}{3} \int_0^{\infty} n(p) p v dp = nkT$$
Homework

- For next class:
  - Problem 4-1