Outline

- Properties of white dwarfs
- White dwarf cooling
- Minimum stellar mass (brown dwarfs)

Equations of Stellar Structure

- Which are valid, relevant for white dwarfs?

$$\frac{dP(r)}{dr} = \frac{-GM(r)\rho(r)}{r^2}$$

$$\frac{dT(r)}{dr} = \frac{-3L(r)\kappa(r)\rho(r)}{16\pi r^2 a c T^3(r)}$$

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r)$$

$$\frac{dL(r)}{dr} = 4\pi r^2 \rho(r)\epsilon(r)$$

Scaling Relations

- Assume $P(r) \propto r^{\beta}$, same for $M(r)$, $\rho(r)$, then $dP/dr \propto P/r$, etc...
- Find scaling relations from equations of stellar structure

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r) \quad \rightarrow \quad M \propto r^3 \rho$$

$$\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2} \quad \rightarrow \quad P \propto \frac{G M \rho}{r} \propto \frac{G M^2}{r^4}$$

- Equation of state for degenerate electron gas

$$P = \left(\frac{3}{\pi}\right)^{2/3} \frac{h^2}{20\, m} m_p^{-5/3} \left(\frac{Z}{A}\right)^{5/3} \rho^{5/3} = b\,\rho^{5/3} \propto b\,\frac{M^{5/3}}{r^5}$$

- Equate pressures to find that radius decreases with mass

$$r \propto \frac{b}{G} M^{-1/3} \qquad r = 2.3 \times 10^9 \text{ cm} \left(\frac{Z}{A}\right)^{5/3} \left(\frac{M}{M_{Sun}}\right)^{-1/3}$$

Limiting mass of White Dwarf

- As mass increases, radius decreases towards zero.

  At what point does this break down?

Pressure

- In calculating pressure, assumed v = p/m.
- This breaks down as $v \rightarrow c$. Instead, $v \approx c$.

$$P \neq \frac{1}{3} \int_0^\infty n(p)\, p\, v\, dp \quad \text{instead} \quad P = \frac{1}{3} \int_0^\infty n(p)\, p\, v\, dp = \frac{8\pi c}{3h^3} \frac{p_f^4}{4}$$

- Then equation of state is

$$P = \left(\frac{3}{8\pi}\right)^{1/3} \frac{hc}{4 m_p^{4/3}} \left(\frac{Z}{A}\right)^{4/3} \rho^{4/3}$$

- Note different exponent on density.
- Why no dependence on mass of electron?

Scaling Relations

- Scaling relations from equations of stellar structure $$P \propto \frac{G M^2}{r^4}$$
- Equation of state for degenerate electron gas is $P \propto \rho^{(4+\varepsilon)/3}$ with $\varepsilon = 1$ for non-relativistic gas, $\varepsilon = 0$ for ultra-relativistic gas.
- Equate pressures to find dependence of radius on mass

$$r \propto M^{(\epsilon-2)/3\epsilon} \qquad \text{as } \epsilon \rightarrow 0, \;\; r \rightarrow M^{-\infty} = 0$$

- When electrons are ultra-relativistic, pressure increases too slowly with density to support the star against collapse as mass increases. Maximum mass, the “Chandrasekhar mass”,

$$M \approx \left(\frac{hc}{G}\right)^{3/2} m_p^{-2} = \left(\frac{hc}{G m_p^2}\right)^{3/2} m_p$$

- Accurately calculated value is 1.4 solar masses.

White Dwarf Cooling

- Thermal energy of star while still supported by thermal pressure is

$$E_{th} \approx \frac{G M^2}{2 r} = \frac{3}{2} N kT \approx \frac{M}{m_p} kT \rightarrow kT \approx \frac{GMm_p}{r}$$

- For *M* and *r* of typical white dwarf, find kT ~ 10 keV, $T \sim 10^8$ K.
- White dwarf radiates as blackbody with it surface temperature.
- Electrons are good conductors in most of star, leading to a near uniform temperature, but a thin layer of normal matter on the surface insulates the star and slows cooling. Ignore this layer to estimate cooling time, which will then be an upper limit.

$$\frac{-dE_{th}}{dt} = L \rightarrow 4\pi r^2 \sigma T^4 = \frac{3}{8} \frac{M k}{m_p} \frac{dT}{dt} \rightarrow dt = \frac{3 M k}{32 \pi \sigma m_p r^2} T^{-4} dT$$

- Solution $T \sim t^{-1/3}$. Cooling time is on order of 10^9 years.

Brown Dwarfs

- As a protostar collapses, fusion turns when the core temperature gets high enough. Hydrogen fusion requires T > 10^7 K.
- We have assumed that radius will keep contracting, until a sufficiently high temperature is reached. But, what is degeneracy pressure stops contraction first?
- Use same "white dwarf" radius calculated before. Now Z/A = 1 since gas is hydrogen.

$$r = 2.3 \times 10^9 \text{ cm} \left(\frac{Z}{A}\right)^{5/3} \left(\frac{M}{M_{Sun}}\right)^{-1/3} \qquad kT \approx \frac{GMm_p}{r} = 10^7 \text{ K}$$

- Equate r and solve for $M = 0.07\ M_{\text{Sun}}$.
- Nuclear fusion in stars of this mass or smaller never turns on. They are "brown dwarfs".

Homework

- For next class:
  - Problem 4-2