Outline

- Go over problem 4-4, exam problem #2
- Spacetime metrics
- Schwarzschild metric
- Black holes
- Binary star systems and tides

General Relativity and the Metric

- Shape of spacetime in general relativity is described by a “metric”.
- The metric is a solution of the Einstein equations (in Chapter 8).
- The metric, $g_{\mu\nu}$, tells how to calculate the “interval” between two nearby spacetime events separated by a four-vector dx_μ .
  - four vector = dx_μ where index μ runs over 4 coordinates, usually one time coordinate and three space coordinates
  - interval = ds, $(ds)^2 = \sum g_{\mu\nu}\, dx_\mu\, dx_\nu$

Flat Space Metric

- In the absence of matter, spacetime is flat.
- Can use 4-vector $dx_\mu = [c \cdot dt, dx, dy, dz]$
- The metric is the Minkowski metric
  - $g_{00} = 1$, $g_{11} = -1$, $g_{22} = -1$, $g_{33} = -1$
- Interval $(ds)^2 = (c \cdot dt)^2 - (dx)^2 - (dy)^2 - (dz)^2$
  - If $dt = 0$, then $(ds)^2 = -[(dx)^2 + (dy)^2 + (dz)^2]$ and $|ds|$ = distance between the two points.
  - If $dx = dy = dz = 0$, then $(ds)^2 = (c \cdot dt)^2$ and ds/c = time between the two points. Proper time $d\tau = ds/c$ is the time that elapse on a clock moved between the two points.
  - Light travels along "null geodesics" for which $ds = 0$.

Schwarzschild Metric

- Can also describe flat spacetime using spherical coordinates
  - 4-vector $dx_\mu = [c \cdot dt, dr, d\theta, d\varphi]$
- The Minkowski metric is then:
  - $g_{00} = 1$, $g_{11} = -1$, $g_{22} = -r^2$, $g_{33} = -r^2 \sin^2\theta$
  - Interval $(ds)^2 = (c \cdot dt)^2 - (dr)^2 - (r \cdot d\theta)^2 - (r \cdot \sin\theta \cdot d\varphi)^2$
- Spacetime surrounding a static, spherically symmetric mass is described by the Schwarzschild metric:

$$(ds)^2 = \left(1 - \frac{2GM}{rc^2}\right)(c\,dt)^2 - \left(1 - \frac{2GM}{rc^2}\right)^{-1}(dr)^2 - (r\,d\theta)^2 - (r\sin\theta\,d\varphi)^2$$

$$(ds)^2 = \left(1 - \frac{r_s}{r}\right)(c\,dt)^2 - \left(1 - \frac{r_s}{r}\right)^{-1}(dr)^2 - (r\,d\theta)^2 - (r\sin\theta\,d\varphi)^2$$

- where $r_s = 2GM/c^2$ = Schwarzschild radius = 3 km (M/M_{Sun})

Event Horizon

- If the mass is so compact that it fits inside it Schwarzschild radius, then it has an “event horizon”, where proper time stops.
- Proper time for clock at rest:

$$d\tau = \frac{ds}{c} = \left(1 - \frac{r_s}{r}\right)^{1/2} dt$$

- As $r \to r_s$ $d\tau \to 0$, gravitational time dilation becomes infinite.
- Gravitational redshift:

$$\frac{\lambda}{\lambda_0} = \left(1 - \frac{r_s}{r}\right)^{-1/2}$$

- where λ_0 = emitted wavelength, λ = observed (at infinity).
- Becomes infinite as $r \to r_s$

Event Horizon

- Coordinate speed of light moving radially = *dr*/*dt*.
- Recall *ds* = 0 for light, so

$$\frac{dr}{dt} = c\left(1-\frac{r_s}{r}\right)$$

  - As $r \rightarrow r_s$ an observer at infinity see *dr/dt* → 0, coordinate speed of light goes to zero → even light cannot propagate out from the event horizon.
- Note that, conversely, collapse of matter to r_s takes an infinite amount of time for an observer at infinity (but a finite amount of time for someone falling in). Effectively, the matter is frozen in time as it fall in. However, there are no observable differences between an object in which the matter is built up just outside r_s and an actual black hole with matter inside r_s.

Tides

- Many stars are in binary systems, about half for solar mass stars, a higher fraction for massive stars.
- Stars in binaries will exert forces on each other.
  - Force on center of mass maintains binary orbit.
  - Force will be stronger for parts of star towards to companion and weaker for parts away from companion – tidal forces.
- Consider mass element, *m*, in star 1 at a distance Δr from the center of star 1. Force on *m* due to star 1:

$$\frac{F_g}{m} = \frac{G M_1}{(\Delta r)^2}$$

- Force on *m* due to star 2 at distance *r*:

$$\frac{F_{tide}}{m} = G M_2 \left(\frac{1}{r^2} - \frac{1}{(r+\Delta r)^2} \right) \approx \frac{2 G M_2 \Delta r}{r^3} \quad \rightarrow \quad \frac{F_{tide}}{F_g} = \frac{2 M_2}{M_1} \left(\frac{\Delta r}{r} \right)^3$$

- Tidal forces are large when $\Delta r/r$ is large, separation ~ size of stars.

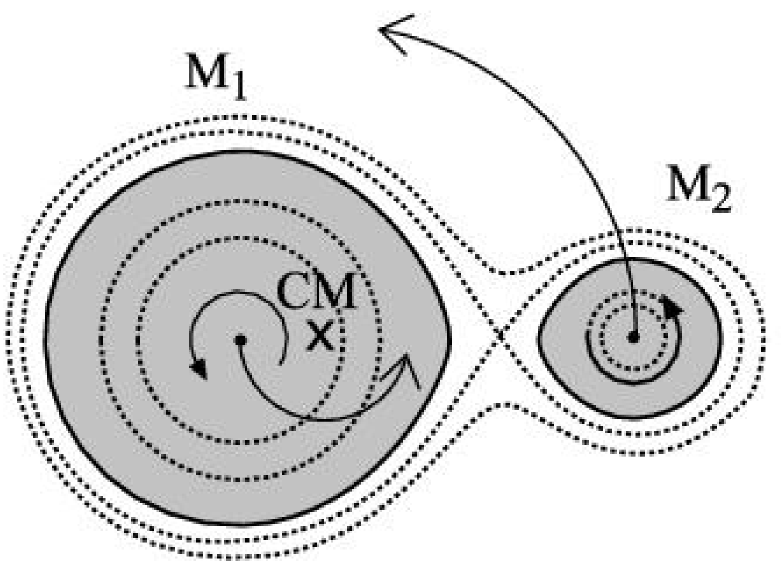
Tides

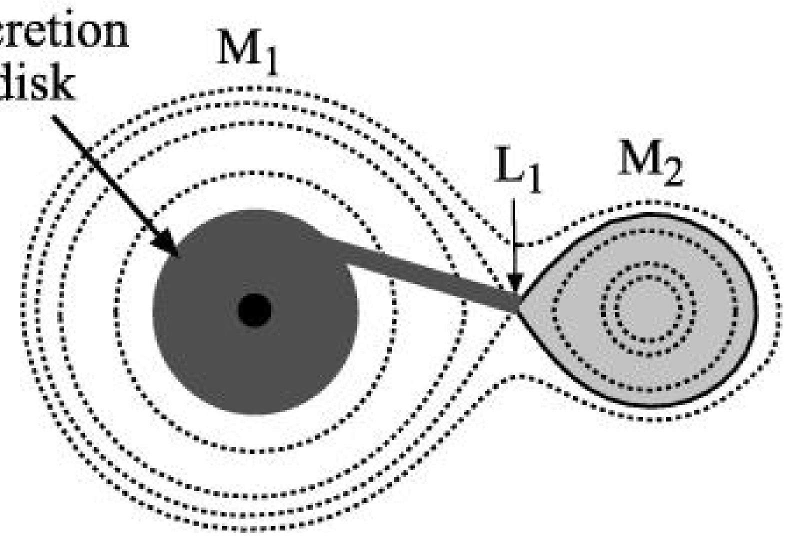
- Tides tend to:
  - convert orbital energy to heat in the stars
  - circularize the orbit
  - align the spin axes of the stars with the orbital plane
  - synchronize the rotation of the stars to the orbit
    - so each star always sees the same face of its companion

- “Tidally locked” systems are stationary in orbit frame.

Tidally Locked Binaries

- Roche lobes are the deepest non-disjoint equipotential surface in the rotating frame (consider gravity and centripetal acceleration).
- Binary systems can be:
  - detached – neither star fills its Roche lobe
  - semi-detached – one star fills its Roche lobe
  - contact – both stars fill their Roche lobes
- If a star fills its Roche lobe, then matter transfers via first Lagrangian point L_1. Matter will have angular momentum and form an accretion disk around the other star.





Homework

- For next class:
  - Problem 4-7
  - Exam problem #3