1) Radiation dominated < 47 kyr
Matter dominated 47 kyr $-$ 9.8 Gyr
Dark energy dominated > 9.8 Gyr

2) Model universe as homogeneous sphere with mass M, radius r. Consider test mass m at surface.
Energy is

$$E = K - V = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

Critical case is dividing line between expanding / collapsing or unbound / bound when $E = 0$

$$E = 0 = \frac{1}{2}mv^2 - \frac{GMm}{r} \Rightarrow v^2 = \frac{2GM}{r} \quad (*)$$

mass of sphere $M = \rho \frac{4}{3}\pi r^3$ ρ = density

velocity of test mass $v = H_0 r$
where H_0 = Hubble constant
plug into $(*)$ $H_0^2 r^2 = \frac{2G}{r}\frac{4}{3}\pi r^3 \rho$

$$\rightarrow H_0^2 = \frac{8\pi}{3}G\rho$$

$$\rho = \frac{3}{8\pi G}H_0^2 = \text{critical density}$$

3) definition of red shift $z = \frac{\lambda_{obs} - \lambda_e}{\lambda_e}$ (*)

λ_e = emitted wavelength

λ_{obs} = observed wavelength

photon wavelength stretches with universe

so $\lambda_{obs} = \frac{\lambda_e}{a}$

for example photon emitted when $a = 0.1$ is observed today when $a=1$ with a wavelength $10\times$ as long.

Plug into (*)

$$z = \frac{\lambda_e/a - \lambda_e}{\lambda_e} = \frac{1}{a} - 1 = z$$

4) $u_r \propto a^{-4} \Rightarrow u_r(a) = u_r(a=1) \cdot a^{-4}$

$$= u_{r,0} \cdot a^{-4}$$

$$z = \frac{1}{a} - 1 \Rightarrow a = \frac{1}{z-1}$$

$$u_r(z=1100) = u_{r,0} \cdot (z-1)^4$$

$$= 0.26 \text{ MeV } m^{-3} \cdot (1100-1)^4$$

$$= 3.8 \times 10^{11} \text{ MeV} \cdot m^{-3}$$

5) $\ddot{a} = H_0^2 \left[-\frac{\Omega_{r,0}}{a^3} - \frac{\Omega_{m,0}}{2a^2} + \Omega_\Lambda a \right]$ $(*)$

cosmological constant dominates so
$\Omega_\Lambda \gg \Omega_r, \Omega_m$

$(*) \rightarrow \quad \ddot{a} = H_0^2 \Omega_\Lambda a \qquad (**)$

try $a = e^{kt}$

$\dot{a} = k e^{kt} \qquad \ddot{a} = k^2 e^{kt}$

plug into $(**)$

$$k^2 e^{kt} = H_0^2 \Omega_\Lambda e^{kt}$$

solution for all t if $k = H_0 \sqrt{\Omega_\Lambda}$

exponential growth since $a = e^{kt}$

time constant $\frac{1}{k} = H_0^{-1} \Omega_\Lambda^{-1/2} \approx 17 \text{Gyr}$

universe expands by a factor $= e^1$
in a time $1/k = 17 \text{Gyr}$