Outline

- Universe is homogenous and isotropic
- Spacetime metrics
- Friedmann-Walker-Robertson metric
- Number of numbers needed to specify a physical quantity.
- Energy-momentum tensor
- Energy-momentum tensor of the universe

Uniformity of the Universe

To do cosmology:

- We assume that the universe is homogeneous and isotropic
  - Matter/energy is distributed evenly
  - Universe is not rotating
  - Laws of physics are same everywhere

- We need a relativistic theory of gravity to describe the universe on very large scales.
- To make the math easier, we assume matter is truly uniformly distributed, i.e. same density at every point in space.
- This means that the spacetime curvature is the same everywhere.

General Relativity and the Metric

- Shape of spacetime in general relativity is described by a “metric”.
- The metric is a solution of the Einstein equations (in Chapter 8).
- The metric, $g_{\mu\nu}$, tells how to calculate the “interval” between two nearby spacetime events separated by a four-vector dx_μ .
  - four vector = dx_μ where index μ runs over 4 coordinates, usually one time coordinate and three space coordinates
  - interval = ds, $(ds)^2 = \sum g_{\mu\nu}\, dx_\mu\, dx_\nu$

Flat Space Metric

- In the absence of matter, spacetime is flat.
- Can use 4-vector $dx_\mu = [c \cdot dt, dx, dy, dz]$
- The metric is the Minkowski metric
  - $g_{00} = 1$, $g_{11} = -1$, $g_{22} = -1$, $g_{33} = -1$
- Interval $(ds)^2 = (c \cdot dt)^2 - (dx)^2 - (dy)^2 - (dz)^2$
  - If $dt = 0$, then $(ds)^2 = -[(dx)^2 + (dy)^2 + (dz)^2]$ and $|ds|$ = distance between the two points.
  - If $dx = dy = dz = 0$, then $(ds)^2 = (c \cdot dt)^2$ and ds/c = time between the two points. Proper time $d\tau = ds/c$ is the time that elapse on a clock moved between the two points.
  - Light travels along "null geodesics" for which $ds = 0$.

Spherical Coordinates

- Can also describe flat spacetime using spherical coordinates
  - 4-vector $dx_\mu = [c \cdot dt, dr, d\theta, d\varphi]$

- The Minkowski metric is then:
  - $g_{00} = 1$, $g_{11} = -1$, $g_{22} = -r^2$, $g_{33} = -r^2\sin^2\theta$
  - Interval $(ds)^2 = (c \cdot dt)^2 - (dr)^2 - (r \cdot d\theta)^2 - (r \cdot \sin\theta \cdot d\varphi)^2$

Friedmann-Robertson-Walker Metric

- Spacetime with constant density and constant curvature is described by the Friedmann-Robertson-Walker metric, interval is:

$$(ds)^2 = c^2 dt^2 - R^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2 \right)$$

- where R = "scale factor", all distances scale with R.
- (r, θ, φ) are "co-moving" coordinates – as R changes as the universe expands, (r, θ, φ) of each galaxy are unchanged.
- Note r is dimensionless.
- k = "curvature parameter",
  - 0 = flat, +1 = hypersphere, -1 = hyperboloid

Proper Distance

- Distance from us to a galaxy at *r*:

$$l = \int_{r=0}^{r} dl = R(t) \int_{r=0}^{r} \frac{dr}{\sqrt{1-kr^2}}$$

- For $k = 1$, gives $l = R \sin^{-1} r$.
- Draw on board.
- Find velocity on board.

Homework

- For now (hand in next class):
  - A light ray is emitted from $r = 0$ at time $t = 0$. Find an expression for $r(t)$ in a flat universe ($k=0$) and in a positively curved universe ($k=1$).

Scalars and Vectors

- Scalar – quantity described by a single number, e.g. mass or speed.
- Vector – quantity that has direction in addition to magnitude, e.g. position, velocity, or momentum.
  - Classical position vector (x, y, z)
  - Relativistic position four vector (ct, x, y, z)
- Why 3 versus 4 components?

Scalars and Vectors

- Why 3 versus 4 components?
  - In classical mechanics can transform positions via rotations and translations. Need 3 components to allow for any possible rotation or translation.

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

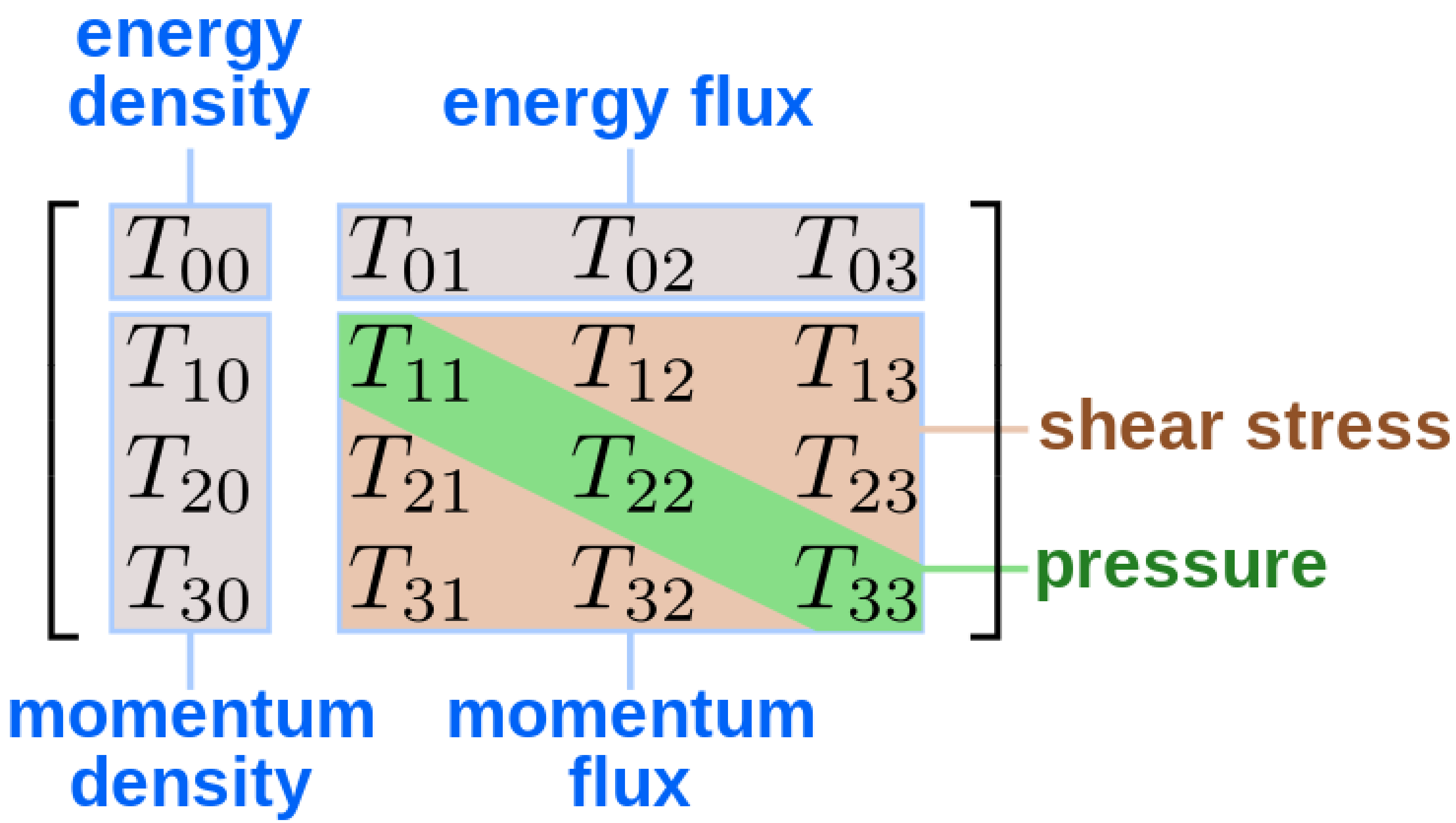
  - In relativistic mechanics, can transform between time and space via Lorentz transform. Need all four components in one vector.

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Fluid

- How many numbers do we need to describe a fluid?
- Look at one cube-shaped region in space. Count up the fluid particles, or better, the total mass/energy in the fluid.
  - At least 1 number.
- Particles could be moving. Need relativistic four momentum vector $(E/c, p_x, p_y, p_z)$ to describe a moving particle.
  - At least 4 numbers.
- Different particles could be moving in different directions. We don't want to have to consider each individual particle, so let's look at net flows through each wall.
  - The *xy*-momentum flux is the amount of *x*-momentum flowing the *y*-direction per unit area and unit time.
  - Note *xx*-momentum flux = pressure along *x*.

Energy-Momentum Tensor



- Need 16 components – a tensor. Also called stress-energy tensor.
- Energy-momentum tensor is symmetric, so only 10 components are independent.
- Momentum flux is also called anisotropic pressure.

Tensor Notation

- Lots of different notations, most common uses super and subscripts.
  - Four-vector have one super/subscript.
  - Rank 2 tensors have two super/subscripts.
  - If same variable appears as a super and subscript, then sum over it.
- Lorentz transform from a few slides back

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

- becomes

$$y_\mu = L_{\mu\nu} x^\nu \quad \text{where} \quad L_{00} = L_{11} = \gamma \quad L_{01} = L_{10} = -\beta\gamma \quad L_{22} = L_{33} = 1$$

Tensor Notation

- Maxwell's equations (the cool way)

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} j^\nu$$

$$j^\nu = (c\rho, j_x, j_y, j_z)$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

Uniformity of the Universe

- We assume that the universe is homogeneous and isotropic
  - there is no anisotropic pressure.
  - there are no net flows
    - Therefore energy-momentum tensor is diagonal.

- Look at first two terms (time, radial) for a co-moving observer
  - T_{00} = energy density = ρc^2, where ρ = mass density
  - $T_{11} = P \cdot g_{11}$ where P = pressure and g_{11} = radial component of metric

$$T_{11} = \frac{P R^2}{1 - kr^2}$$

Homework

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