Outline

- Universe is homogenous and isotropic
- Spacetime metrics
- Friedmann-Walker-Robertson metric
- Number of numbers needed to specify a physical quantity.
- Energy-momentum tensor
- Energy-momentum tensor of the universe
Uniformity of the Universe

To do cosmology:

- We assume that the universe is homogeneous and isotropic
  - Matter/energy is distributed evenly
  - Universe is not rotating
  - Laws of physics are same everywhere

- We need a relativistic theory of gravity to describe the universe on very large scales.

- To make the math easier, we assume matter is truly uniformly distributed, i.e. same density at every point in space.

- This means that the spacetime curvature is the same everywhere.
General Relativity and the Metric

- Shape of spacetime in general relativity is described by a “metric”.
- The metric is a solution of the Einstein equations (in Chapter 8).
- The metric, $g_{\mu\nu}$, tells how to calculate the “interval” between two nearby spacetime events separated by a four-vector $dx_\mu$.
  - four vector = $dx_\mu$ where index $\mu$ runs over 4 coordinates, usually one time coordinate and three space coordinates
  - interval = $ds$, $(ds)^2 = \sum g_{\mu\nu} dx_\mu dx_\nu$
Flat Space Metric

- In the absence of matter, spacetime is flat.
- Can use 4-vector $\mathbf{dx}_\mu = [c \cdot dt, dx, dy, dz]$
- The metric is the Minkowski metric
  - $g_{00} = 1, \ g_{11} = -1, \ g_{22} = -1, \ g_{33} = -1$
- Interval $(ds)^2 = (c \cdot dt)^2 - (dx)^2 - (dy)^2 - (dz)^2$
  - If $dt = 0$, then $(ds)^2 = -[(dx)^2 + (dy)^2 + (dz)^2]$ and $|ds| = \text{distance between the two points}$.
  - If $dx = dy = dz = 0$, then $(ds)^2 = (c \cdot dt)^2$ and $ds/c = \text{time between the two points}$. Proper time $d\tau = ds/c$ is the time that elapse on a clock moved between the two points.
  - Light travels along “null geodesics” for which $ds = 0$. 
Spherical Coordinates

- Can also describe flat spacetime using spherical coordinates
  - 4-vector \( \mathbf{dx}_\mu = [c \cdot dt, dr, d\theta, d\phi] \)

- The Minkowski metric is then:
  - \( g_{00} = 1, \quad g_{11} = -1, \quad g_{22} = -r^2, \quad g_{33} = -r^2 \sin^2 \theta \)
  - Interval \((ds)^2 = (c \cdot dt)^2 - (dr)^2 - (r \cdot d\theta)^2 - (r \cdot \sin \theta \cdot d\phi)^2\)
Friedmann-Robertson-Walker Metric

- Spacetime with constant density and constant curvature is described by the Friedmann-Robertson-Walker metric, interval is:

\[(ds)^2 = c^2 dt^2 - R^2 \left( \frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \right)\]

- where \( R \) = “scale factor”, all distances scale with \( R \).
- \((r, \theta, \varphi)\) are “co-moving” coordinates – as \( R \) changes as the universe expands, \((r, \theta, \varphi)\) of each galaxy are unchanged.
- Note \( r \) is dimensionless.
- \( k \) = “curvature parameter”,
  - 0 = flat, +1 = hypersphere, -1 = hyperboloid
Proper Distance

- Distance from us to a galaxy at \( r \):

\[
l = \int_{r=0}^{r} dl = R(t) \int_{r=0}^{r} \frac{dr}{\sqrt{1-kr^2}}
\]

- For \( k = 1 \), gives \( l = R \sin^{-1} r \).

- Draw on board.

- Find velocity on board.
Homework

• For now (hand in next class):
  - A light ray is emitted from \( r = 0 \) at time \( t = 0 \). Find an expression for \( r(t) \) in a flat universe \((k=0)\) and in a positively curved universe \((k=1)\).
Scalars and Vectors

- Scalar – quantity described by a single number, e.g. mass or speed.
- Vector – quantity that has direction in addition to magnitude, e.g. position, velocity, or momentum.
  - Classical position vector \((x, y, z)\)
  - Relativistic position four vector \((ct, x, y, z)\)
- Why 3 versus 4 components?
Scalars and Vectors

- Why 3 versus 4 components?
  
  - In classical mechanics can transform positions via rotations and translations. Need 3 components to allow for any possible rotation or translation.
    
    \[
    \begin{pmatrix}
    x' \\
    y' \\
    z'
    \end{pmatrix}
    =
    \begin{pmatrix}
    R_{xx} & R_{xy} & R_{xz} \\
    R_{yx} & R_{yy} & R_{yz} \\
    R_{zx} & R_{zy} & R_{zz}
    \end{pmatrix}
    \begin{pmatrix}
    x \\
    y \\
    z
    \end{pmatrix}
    \]

  - In relativistic mechanics, can transform between time and space via Lorentz transform. Need all four components in one vector.
    
    \[
    \begin{pmatrix}
    ct' \\
    x' \\
    y' \\
    z'
    \end{pmatrix}
    =
    \begin{pmatrix}
    \gamma & -\beta \gamma & 0 & 0 \\
    -\beta \gamma & \gamma & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1
    \end{pmatrix}
    \begin{pmatrix}
    ct \\
    x \\
    y \\
    z
    \end{pmatrix}
    \]
Fluid

- How many numbers do we need to describe a fluid?
- Look at one cube-shaped region in space. Count up the fluid particles, or better, the total mass/energy in the fluid.
  - At least 1 number.
- Particles could be moving. Need relativistic four momentum vector \((E/c, p_x, p_y, p_z)\) to describe a moving particle.
  - At least 4 numbers.
- Different particles could be moving in different directions. We don't want to have to consider each individual particle, so let's look at net flows through each wall.
  - The xy-momentum flux is the amount of x-momentum flowing the y-direction per unit area and unit time.
  - Note xx-momentum flux = pressure along x.
Energy-Momentum Tensor

- Need 16 components – a tensor. Also called stress-energy tensor.
- Energy-momentum tensor is symmetric, so only 10 components are independent.
- Momentum flux is also called anisotropic pressure.
Tensor Notation

- Lots of different notations, most common uses super and subscripts.
  - Four-vector have one super/subscript.
  - Rank 2 tensors have two super/subscripts.
  - If same variable appears as a super and subscript, then sum over it.

- Lorentz transform from a few slides back

\[
\begin{pmatrix}
ct' \\
x' \\
y' \\
z'
\end{pmatrix} =
\begin{pmatrix}
\gamma & -\beta \gamma & 0 & 0 \\
-\beta \gamma & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
ct \\
x \\
y \\
z
\end{pmatrix}
\]

- becomes

\[y_\mu = L_{\mu \nu} x^\nu \quad \text{where} \quad L_{00} = L_{11} = \gamma \quad L_{01} = L_{10} = -\beta \gamma \quad L_{22} = L_{33} = 1\]
Tensor Notation

- Maxwell's equations (the cool way)

\[
\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} j^\nu
\]

\[
J^\nu = (c\rho, j_x, j_y, j_z)
\]

\[
F^{\mu\nu} = \begin{pmatrix}
0 & -E_x & -E_y & -E_z \\
E_x & 0 & -B_z & B_y \\
E_y & B_z & 0 & -B_x \\
E_z & -B_y & B_x & 0
\end{pmatrix}
\]
Uniformity of the Universe

- We assume that the universe is homogeneous and isotropic
  - there is no anisotropic pressure.
  - there are no net flows
    - Therefore energy-momentum tensor is diagonal.

- Look at first two terms (time, radial) for a co-moving observer
  - $T_{00}$ = energy density = $\rho c^2$, where $\rho$ = mass density
  - $T_{11} = P \cdot g_{11}$ where $P$ = pressure and $g_{11}$ = radial component of metric
    \[
    T_{11} = \frac{P R^2}{1 - kr^2}
    \]
Homework

• For next class:
  
  - A light ray is emitted from $r = 0$ at time $t = 0$. Find an expression for $r(t)$ in a flat universe ($k=0$) and in a positively curved universe ($k=1$).