Outline

- Go over homework
- Einstein equations
- Friedmann equations
Homework

• For next class:
  – A light ray is emitted from \( r = 0 \) at time \( t = 0 \). Find an expression for \( r(t) \) in a flat universe (\( k=0 \)) and in a positively curved universe (\( k=1 \)).
Einstein Equations

• General relativity describes gravity in terms of curved spacetime.
• Mass/energy causes the curvature
  – Described by mass-energy tensor
  – Which, in general, is a function of position
• The metric describes the spacetime
  – Metric determines how particles will move

• To go from Newtonian gravity to General Relativity

\[ \frac{d^2 \hat{r}}{dt^2} = \frac{GM}{r^2} \hat{r} \rightarrow G_{\mu \nu}(g_{\mu \nu}) = \frac{8 \pi G}{c^4} T_{\mu \nu} \]

• Where G is some function of the metric
Curvature

- How to calculate curvature of spacetime from metric?
  - “readers unfamiliar with tensor calculus can skip down to Eq. 8.22”
  - There are no equations with tensors after 8.23

- Curvature of spacetime is quantified using parallel transport

  - Hold a vector and keep it parallel to your direction of motion as you move around on a curved surface.

  - Locally, the vector stays parallel between points that are close together, but as you move finite distances the vector rotates – in a manner depending on the path.
Curvature

- Mathematically, one uses “Christoffel symbols” to calculate the “affine connection” which is how to do parallel transport.

- Christoffel symbols involve derivatives of the metric.

\[
\Gamma^\mu_{\sigma\nu} = \frac{1}{2} g^{\mu\rho} \left( \frac{\partial g_{\sigma\rho}}{\partial x^\nu} + \frac{\partial g_{\nu\rho}}{\partial x^\sigma} + \frac{\partial g_{\sigma\nu}}{\partial x^\rho} \right)
\]

- The Riemann tensor measures “the extent to which the metric tensor is not locally isometric to that of Euclidean space”.

\[
R^\alpha_{\beta\gamma\delta} = \frac{\partial \Gamma^\alpha_{\beta\delta}}{\partial x^\gamma} - \frac{\partial \Gamma^\alpha_{\beta\gamma}}{\partial x^\delta} + \Gamma^\alpha_{\rho\gamma} \Gamma^\rho_{\beta\delta} - \Gamma^\alpha_{\rho\delta} \Gamma^\rho_{\beta\gamma}
\]

- The Ricci tensor measures how much a geodesic ball deviates a ball in flat space, also comes in scalar (book is incorrect).

\[
R_{\beta\gamma} = R^\alpha_{\beta\alpha\gamma} = g^{\alpha\delta} R_{\alpha\beta\delta\gamma} = R = g^{\beta\gamma} R_{\beta\gamma}
\]
Einstein Equations

- Once you have all those, you can finally define the Einstein tensor or the “trace-reversed Ricci tensor”:

\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \]

- Note that fundamentally, the Einstein tensor is only a function of the metric.

- The Einstein equations then give the relation between the Einstein tensor and the mass-energy tensor:

\[ G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \]
Friedmann-Robertson-Walker Metric

- Spacetime with constant density and constant curvature is described by the Friedmann-Robertson-Walker metric, interval is:

\[
(ds)^2 = c^2 dt^2 - R^2 \left( \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right)
\]

- We found that the mass-energy tensor in a FRW universe is diagonal, thus \( G_{\mu\nu} \) must also be diagonal.

\[
G_{00} = \frac{3}{c^2 R^2} \left( \frac{dR}{dt} + \frac{k}{c^2} R \right)
\]

\[
G_{11} = -\frac{2 \frac{dR}{dt} + \frac{d^2 R}{dt^2} + kc^2}{c^2 (1 - kr^2)}
\]

- Recall

\[
T_{00} = \rho c^2
\]

\[
T_{11} = \frac{PR^2}{1 - kr^2}
\]

- Derive Friedmann equations on the board
Uniformity of the Universe

- We assume that the universe is homogeneous and isotropic
  - there is no anisotropic pressure.
  - there are no net flows
    - Therefore energy-momentum tensor is diagonal.

- Look at first two terms (time, radial) for a co-moving observer
  - $T_{00}$ = energy density = $\rho c^2$, where $\rho$ = mass density
  - $T_{11} = P \cdot g_{11}$ where $P$ = pressure and $g_{11}$ = radial component of metric

\[
T_{11} = \frac{PR^2}{1-kr^2}
\]
Homework

• For next class:
  – What does it mean that the Universe is expanding?