Outline

- Go over homework
- Einstein equations
- Friedmann equations

Homework

- For next class:
  - A light ray is emitted from $r = 0$ at time $t = 0$. Find an expression for $r(t)$ in a flat universe ($k=0$) and in a positively curved universe ($k=1$).

Einstein Equations

- General relativity describes gravity in terms of curved spacetime.
- Mass/energy causes the curvature
  - Described by mass-energy tensor
  - Which, in general, is a function of position
- The metric describes the spacetime
  - Metric determines how particles will move

- To go from Newtonian gravity to General Relativity

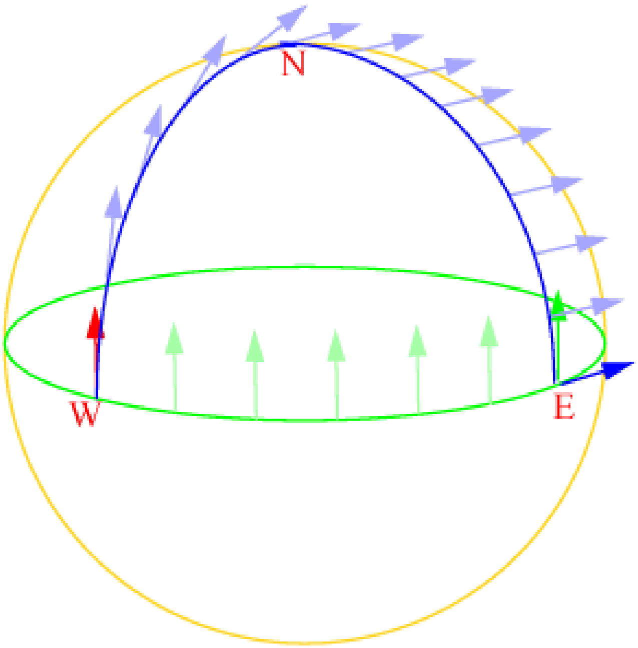
$$\frac{d^2}{dt^2}\vec{r} = \frac{GM}{r^2}\hat{r} \rightarrow G_{\mu\nu}(g_{\mu\nu}) = \frac{8\pi G}{c^4}T_{\mu\nu}$$

- Where G is some function of the metric

Curvature

- How to calculate curvature of spacetime from metric?
  - "readers unfamiliar with tensor calculus can skip down to Eq. 8.22"
  - There are no equations with tensors after 8.23

- Curvature of spacetime is quantified using parallel transport



Path dependency of Parallel Transport

- Hold a vector and keep it parallel to your direction of motion as you move around on a curved surface.
- Locally, the vector stays parallel between points that are close together, but as you move finite distances the vector rotates – in a manner depending on the path.

Curvature

- Mathematically, one uses “Christoffel symbols” to calculate the “affine connection” which is how to do parallel transport.
- Christoffel symbols involve derivatives of the metric.

$$\Gamma^{\mu}_{\sigma\nu} = \frac{1}{2} g^{\mu\rho} \left(\frac{\partial g_{\sigma\rho}}{\partial x^{\nu}} + \frac{\partial g_{\nu\rho}}{\partial x^{\sigma}} + \frac{\partial g_{\sigma\nu}}{\partial x^{\rho}} \right)$$

- The Reimann tensor measures “the extent to which the metric tensor is not locally isometric to that of Euclidean space”.

$$R^{\alpha}_{\beta\gamma\delta} = \frac{\partial \Gamma^{\alpha}_{\beta\delta}}{\partial x^{\gamma}} - \frac{\partial \Gamma^{\alpha}_{\beta\gamma}}{\partial x^{\delta}} + \Gamma^{\alpha}_{\rho\gamma} \Gamma^{\rho}_{\beta\delta} - \Gamma^{\alpha}_{\rho\delta} \Gamma^{\rho}_{\beta\gamma}$$

- The Ricci tensor measures how much a geodesic ball deviates a ball in flat space, also comes in scalar (book is incorrrect).

$$R_{\beta\gamma} = R^{\alpha}_{\beta\alpha\gamma} = g^{\alpha\delta} R_{\alpha\beta\delta\gamma} \qquad R = g^{\beta\gamma} R^{\beta\gamma}$$

Einstein Equations

- Once you have all those, you can finally define the Einstein tensor or the " trace-reversed Ricci tensor":

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

  - Note that fundamentally, the Einstein tensor is only a function of the metric.

- The Einstein equations then give the relation between the Einstein tensor and the mass-energy tensor

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Friedmann-Robertson-Walker Metric

- Spacetime with constant density and constant curvature is described by the Friedmann-Robertson-Walker metric, interval is:

$$(ds)^2 = c^2 dt^2 - R^2 \left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2 \right)$$

- We found that the mass-energy tensor in a FRW universe is diagonal, thus $G_{\mu\nu}$ must also be diagonal.

$$G_{00} = \frac{3}{c^2 R^2}\left(\dot{R}^2 + kc^2\right) \qquad G_{11} = -\frac{2R\ddot{R} + \dot{R}^2 + kc^2}{c^2\left(1-kr^2\right)}$$

- Recall

$$T_{00} = \rho c^2 \qquad T_{11} = \frac{PR^2}{1-kr^2}$$

- Derive Friedmann equations on the board

Uniformity of the Universe

- We assume that the universe is homogeneous and isotropic
  - there is no anisotropic pressure.
  - there are no net flows
    - Therefore energy-momentum tensor is diagonal.

- Look at first two terms (time, radial) for a co-moving observer
  - T_{00} = energy density = ρc^2, where ρ = mass density
  - $T_{11} = P \cdot g_{11}$ where P = pressure and g_{11} = radial component of metric

$$T_{11} = \frac{P R^2}{1 - kr^2}$$

Homework

- For next class:
  - What does it mean that the Universe is expanding?