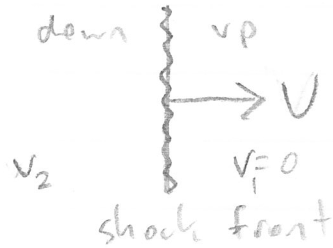
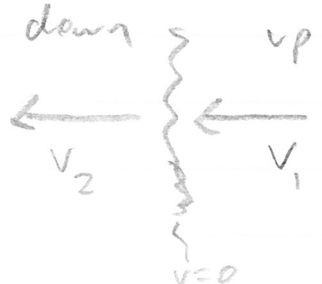
1^{st} order Fermi acceleration
plane shock front

observer's frame



look at frame where shock is at rest



$$\rho_1 v_1 = \rho_2 v_2$$

strong shock

$$\frac{\rho_2}{\rho_1} = \frac{\gamma+1}{\gamma-1}$$

γ = ratio of specific heats

take $\gamma = \frac{5}{3}$ = monoatomic gas

$$\frac{\rho_2}{\rho_1} = 4 \Rightarrow v_2 = \frac{1}{4} v_1$$

look at from frame of up stream gas

$\frac{3}{4}U \longrightarrow$ stationary

particle passing from up to down stream has head-on collision

frame of down stream gas

stationary $\longleftarrow \frac{3}{4}U$

particle passing from down to up has head on collision

Energy increase on one passage

$\gamma = \gamma(V) = \dfrac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

$$E' = \gamma\,(E + Vp\cos\theta)$$

assume relativistic particles $\Rightarrow$ $E \approx pc$

$$E' = \gamma\left(E + V\frac{E}{c}\cos\theta\right)$$

$\frac{v}{c} = 0.5 \rightarrow \gamma = 1.2$

take $\gamma \sim 1$ = non-rel shock

$\frac{v}{c} = 0.866 \rightarrow \gamma = 2$

$$\frac{\Delta E}{E} = \frac{V}{c}\cos\theta$$

probability of crossing shock

$$p(\theta) = 2 \sin\theta \cos\theta \, d\theta \qquad 0 \leq \theta \leq \frac{\pi}{2}$$

$$\left\langle \frac{\Delta E}{E} \right\rangle = \int \frac{\Delta E(\theta)}{E} p(\theta) \, d\theta$$

$$= \int_0^{\pi/2} \frac{V}{c} \cos\theta \; 2 \sin\theta \cos\theta \, d\theta = \frac{2}{3} \frac{V}{c}$$

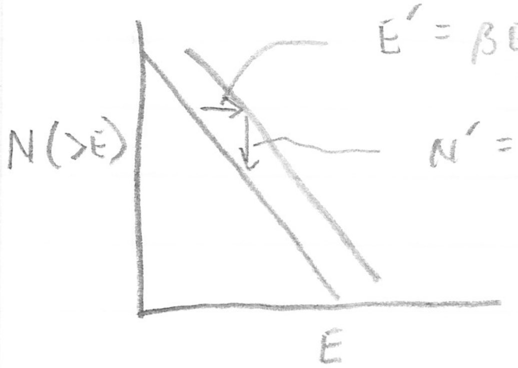
idea is that particle crosses shock and gains energy,
velocity is then randomized by elastic collisions,
particle then crosses shock in opposite direction

In one round trip

$$\left\langle \frac{\Delta E}{E} \right\rangle = \frac{4}{3} \frac{V}{c}$$

$$V = \frac{3}{4} U \Rightarrow \left\langle \frac{\Delta E}{E} \right\rangle = \frac{U}{c}$$

Find distribution N vs E



$E' = \beta E$ β = energy increase

$N' = PN$ P = probability that particle does not escape

$$N'(>E) = N\left(>\frac{E}{\beta}\right) P$$

want stationary solution $N'(>E) = N(>E)$

try $N = N_0 E^a$

$$N_0 E^a = N_0 E^a \beta^{-a} P$$

$$\frac{1}{P} = \beta^{-a}$$

$$-\ln P = -a \ln \beta \Rightarrow a = \frac{\ln P}{\ln \beta}$$

$$N(>E) = N_0 E^a$$

$$\frac{dN}{dE} = a N_0 E^{a-1}$$

$$N(E)\, dE \propto E^{-1 + \ln P / \ln \beta}\, dE$$

later

For interactions with randomly moving clouds need know density of clouds, cross section for interactions, escape time, ~~ not universal

Fermi initially suggested CR are accelerated by clouds in ISM but this doesn't work because there are not enough clouds.

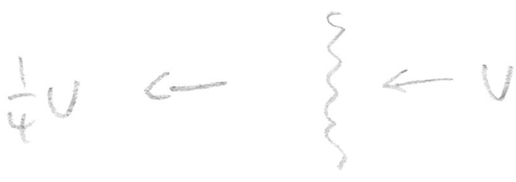
For strong shocks

$$\beta = 1 + \frac{U}{c}$$

Rate of particles crossing shock $= \frac{1}{4} N c$

$N =$ number density of particles

In shock frame



particles are carried away from shock at rate $\frac{1}{4} U N$ assumes CR move with gas

so $\dfrac{\text{rate loss}}{\text{rate crossing}} = \dfrac{\frac{1}{4} U N}{\frac{1}{4} c N} = \dfrac{U}{c}$ $\quad P = 1 - \dfrac{U}{c}$

$$N(E) \propto E^{-1 + \ln P / \ln \beta}$$

$$\cancel{\text{exponent} = -1 + \frac{\ln 1 - \frac{U}{c}}{\quad}}$$

$$\ln P = \ln\left(1 - \frac{U}{c}\right) \approx -\frac{U}{c}$$

$$\ln \beta = \ln\left(1 + \frac{U}{c}\right) \approx \frac{U}{c}$$

$$\text{exponent} = -1 + \frac{-U/c}{U/c} = -1 - 1 = -2$$

$$\Rightarrow N(E) \propto E^{-2}$$

Maximum energy

$$\frac{\Delta E}{E} = \frac{U}{c} = \xi$$

$$\frac{dE}{dt} = \frac{\xi E}{T_{cycle}}$$

$T_{cycle} = ?$

Lamour radius $r_g = \frac{pc}{ZeB} = \frac{E}{ZeB}$

Time it takes particle to drift across shock?

$$T_{cycle} \sim \frac{r_g}{U} \Rightarrow \frac{20}{3} \frac{E}{U ZeB} = T_{cycle}$$

$$E_{max} = \int_0^T \frac{dE}{dt} dt = \int \frac{\xi E}{T_{cycle}} dt = \frac{3}{20} \frac{U^2 ZeB}{c} T$$

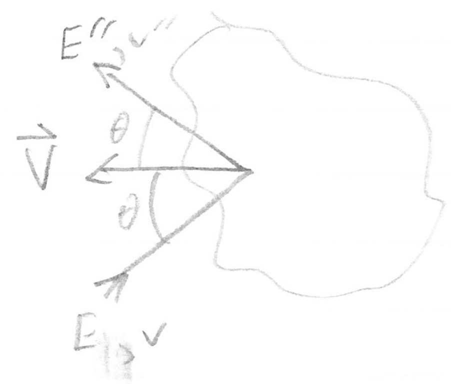
T = how long SN shock can efficiently accelerate particles = free expansion phase

$\sim$ 1000 years

slow SN 1006

for $B \sim 3 \mu G \Rightarrow E_{max} \sim Z \times 30$ TeV

2^{nd} order
Fermi acceleration
charged particle bounces off $\vec{B}$ in moving cloud



assume elastic collision
mass cloud >> mass particle

find ΔE by considering bounce in cloud reference frame $p_x \to -p_x$

Transform to cloud frame
bounce
transform back

$$E'' = \gamma^2 E \left(1 + \frac{2Vv\cos\theta}{c^2} + \frac{V^2}{c^2} \right)$$

$$\frac{\Delta E}{E} = \frac{2Vv\cos\theta}{c^2} + 2\frac{V^2}{c^2}$$

2^{nd} order Fermi acceleration
consider a large number of random bounces

$$\langle \Delta E \rangle = \int \Delta E(\theta)\, p(\theta)\, d\theta$$

$$p(\theta) \propto \gamma \left(1 + \frac{V}{c} \cos\theta\right) \sin\theta \, d\theta \quad \text{for } v \approx c$$

$$\langle \cos\theta \rangle = \frac{\int (1+\beta\cos\theta) \cos\theta \sin\theta \, d\theta}{\int (1+\beta\cos\theta) \sin\theta \, d\theta}$$

$$x = \cos\theta \qquad dx = \sin\theta \, d\theta$$

$$\langle \cos\theta \rangle = \frac{\int_{-1}^{1} (1+\beta x)\, x \, dx}{\int_{-1}^{1} (1+\beta x)\, dx} = \frac{1}{3}\beta = \frac{V}{3c}$$

$$\left\langle \frac{\Delta E}{E} \right\rangle = \frac{2Vv}{c^2} \langle \cos\theta \rangle + 2\frac{V^2}{c^2}$$

$$= \frac{2Vc}{c^2} \frac{V}{3c} + \frac{2V^2}{c^2} = \frac{8}{3}\left(\frac{V}{c}\right)^2$$

energy increase in second order in $\frac{V}{c}$

support

$$\langle \cos\theta \rangle = \frac{\int (1+\beta\cos\theta)\cos\theta \sin\theta \, d\theta}{\int (1+\beta\cos\theta)\sin\theta \, d\theta}$$

$x = \cos\theta \qquad dx = \sin\theta \, d\theta$

$$\langle \cos\theta \rangle = \frac{\int_{-1}^{1} (1+\beta x)\, x \, dx}{\int_{-1}^{1} (1+\beta x)\, dx} = \frac{1}{3}\beta$$

$$\int_{-1}^{1} x + \beta x^2 \, dx = \frac{x^2}{2}\Big|_{x=-1}^{1} + \frac{\beta}{3}x^3\Big|_{x=-1}^{1} = \frac{2\beta}{3}$$

$$\int_{-1}^{1} 1 + \beta x \, dx = x\Big|_{-1}^{1} + \beta\frac{x^2}{2}\Big|_{x=-1}^{1} = 2$$

$$\langle \cos\theta \rangle = \frac{\frac{2\beta}{3}}{2} = \frac{1}{3}\beta$$