First order Fermi acceleration plane shock front

Observer frame

down \rightarrow v_p

\rightarrow U

v_2

v_f = 0

Shock front

Look at frame where shock is at rest

down \rightarrow v_p

\leftarrow v_1

v_f = 0

\rho_1 v_1 = \rho_2 v_2

Strong shock

\frac{\rho_2}{\rho_1} = \frac{\gamma+1}{\gamma-1}

\gamma = \text{ratio of specific heats}

take \ \gamma = \frac{5}{3} \ \text{= monoatomic gas}

\frac{P_2}{P_1} = 4 \ \Rightarrow \ v_2 = \frac{1}{4} v_1

\frac{1}{4} U \leftarrow \rightarrow U
look at from frame of up stream gas

\[ \frac{3}{4} U \rightarrow \begin{cases} \text{stationary} \\
\text{stationary} \end{cases} \]

particle passing from up to down stream has head-on collision

frame of down stream gas

\[ \begin{cases} \text{stationary} \\
\text{stationary} \end{cases} \leftarrow \frac{3}{4} U \]

particle passing from down to up has head-on collision

Energy increase on one passage

\[ \delta = \delta(V) \]

\[ E' = \gamma \left( E + V \rho \cos \theta \right) \]

assumed relativistic particles \( \Rightarrow \) \( E \approx pc \)

\[ E' = \gamma \left( E + V \frac{E}{c} \cos \theta \right) \]

take \( \gamma \approx 1 \) non-rel. shock

\[ \frac{\Delta E}{E} = \frac{V}{c} \cos \theta \]
probability of crossing shock

\[ p(\theta) = 2 \sin \theta \cos \theta \, d\theta \quad 0 \leq \theta \leq \frac{\pi}{2} \]

\[ \langle \frac{\Delta E}{E} \rangle = \int \frac{\Delta E(\theta)}{E} \, p(\theta) \, d\theta \]

\[ = \int_{0}^{\pi/2} \frac{V}{c} \cos \theta \, 2 \sin \theta \cos \theta \, d\theta = \frac{2}{3} \frac{V}{c} \]

The idea is that a particle crosses shock and gains energy, velocity is then randomized by elastic collisions, particle then crosses shock in opposite direction.

In one round trip

\[ \langle \frac{\Delta E}{E} \rangle = \frac{V}{3} \frac{V}{c} \]

\[ V = \frac{3}{4} U \Rightarrow \langle \frac{\Delta E}{E} \rangle = \frac{U}{c} \]
Find distribution $N$ vs $E$

$E' = \beta E$

$\beta = \text{energy increase}$

$P = \text{probability}$

that particle does not escape

$N' = PN$

$P = \text{probability}$

that particle does not escape

$N' (>E) = N(\frac{E}{\beta}) P$

want stationary solution $N' (>E) = N (>E)$

try $N = N_0 E^a$

$N_0 E^a = N_0 E^a \beta^{-a} P$

$\frac{1}{P} = \beta^{-a}$

$-\ln P = -a \ln \beta \implies a = \frac{\ln P}{\ln \beta}$

$N(>E) = N_0 E^a$

$\frac{dN}{dE} = a N_0 E^{a-1}$

$N(E) dE \propto E^{-1 + \ln P/\ln \beta} dE$
For interactions with randomly moving clouds, need known density of clouds, cross section for interactions, escape time, not universal.

Fermi initially suggested CR are accelerated by clouds in ISM but this doesn't work because there are not enough clouds.

For strong shocks
\[ \beta = 1 + \frac{U}{c} \]

Rate of particles crossing shock = \( \frac{1}{4} N c \)

\( N \) = number density of particles

In shock frame

\[ \frac{1}{4} U < S < U \]

Particles are carried away from shock at rate \( \frac{1}{4} U N \), assumes CR move with gas.

So, rate loss \( \text{rate crossing} = \frac{1}{2} c N = \frac{U}{c} \)

\[ \rho = 1 - \frac{U}{c} \]
\[ N(E) \propto E^{-1 + \ln \beta / \ln \beta} \]

\[
\ln \beta = \ln (1 + \frac{U}{c}) \approx \frac{U}{c} \\
\ln \beta = \ln (1 + \frac{U}{c}) \approx \frac{U}{c} \\
\text{exponent} = -1 + \frac{-U/c}{U/c} = -1 -1 = -2
\]

\[ \Rightarrow N(E) \propto E^{-2} \]
Maximum energy

\[
\frac{\Delta E}{E} = \frac{U}{c} = \xi
\]

\[
\frac{dE}{dt} = \frac{\xi E}{T_{\text{cycle}}}
\]

\[T_{\text{cycle}} = ?\]

Larmor radius \( r_0 = \frac{p_c}{2eB} = \frac{E}{2eB} \)

Time it takes particle to drift across shock:

\[T_{\text{cycle}} \sim \frac{r_0}{U} \Rightarrow \frac{20}{3} \frac{E}{U} \frac{E}{2eB} = T_{\text{cycle}} \]

\[E = \min \left( \int_0^T \frac{\xi E}{T_{\text{cycle}}} \, dt = \frac{3}{20} \frac{U^2}{c} \right) T \]

\( T = \) how long SN shock can efficiently accelerate particles \( = \) free expansion phase

\( \sim 1000 \text{ years} \)

for \( B \sim 3 \mu B \Rightarrow E_{\text{max}} \sim 2 \times 30 \text{ TeV} \)
2nd order

Fermi acceleration
charged particle bounces off \( B \) in moving cloud

\[
\begin{align*}
\vec{E} & \ ightarrow \vec{E}
\end{align*}
\]

assume elastic collision
mass cloud \( \gg \) mass particle

find \( \Delta E \) by considering bounce
in cloud reference frame \( p_x \rightarrow -p_x \)

Transform to cloud frame
bounce
transform back

\[
E'' = \beta^2 E \left(1 + \frac{2 V v \cos \theta}{c^2} + \frac{V^2}{c^2}\right)
\]

\[
\frac{\Delta E}{E} = \frac{2 V v \cos \theta}{c^2} + 2 \frac{V^2}{c^2}
\]
$2^{nd}$ order Fermi acceleration consider a large number of random

$$\langle \Delta E \rangle = \int \Delta E(\theta) p(\theta) \, d\theta$$

$$p(\theta) = \frac{8}{c} \left(1 + \frac{V}{c} \cos \theta \right) \sin \theta \, d\theta \quad \text{for} \quad V \ll c$$

$$\langle \cos \theta \rangle = \frac{\int \left(1 + \beta \cos \theta \right) \cos \theta \, \sin \theta \, d\theta}{\int \left(1 + \beta \cos \theta \right) \sin \theta \, d\theta}$$

$$x = \cos \theta \quad dx = \sin \theta \, d\theta$$

$$\langle \cos \theta \rangle = \frac{\int_{-1}^{1} \left(1 + \beta x \right) x \, dx}{\int_{-1}^{1} \left(1 + \beta x \right) \, dx} = \frac{1}{3} \beta = \frac{V}{3c}$$

$$\frac{\Delta E}{E} = \frac{2Vv}{c^2} \langle \cos \theta \rangle + 2 \frac{V^2}{c^2}$$

$$= \frac{2Vc}{c^2} \frac{V}{3c} + 2 \frac{V^2}{c^2} = \frac{8}{3} \left( \frac{V}{c} \right)^2$$

energy increase in second order in $\frac{V}{c}$
\[ \langle \cos \theta \rangle = \frac{\int (1 + \beta \cos \theta) \cos \theta \sin \theta \, d\theta}{\int (1 + \beta \cos \theta) \sin \theta \, d\theta} \]

\[ x = \cos \theta \quad dx = \sin \theta \, d\theta \]

\[ \langle \cos \theta \rangle = \frac{\int_{-1}^{1} (1 + \beta x) \times dx}{\int_{-1}^{1} (1 + \beta x) \, dx} = \frac{1}{3} \beta \]

\[ \int_{-1}^{1} x + \beta x^2 \, dx = \left. \frac{x^2}{2} \right|_{x=-1}^{x=1} + \left. \frac{\beta x^3}{3} \right|_{x=-1}^{x=1} = \frac{2\beta}{3} \]

\[ \int_{-1}^{1} 1 + \beta x \, dx = \left. x \right|_{x=-1}^{x=1} + \beta \left. \frac{x^2}{2} \right|_{x=-1}^{x=1} = 2 \]

\[ \langle \cos \theta \rangle = \frac{\frac{2\beta}{3}}{2} = \frac{1}{3} \beta \]