Study suggestions

- memorize det for 2×2, 3×3
- remember $(-1)^{i+j}$ in calculating inverse
- practice inverse for 2×2, 3×3
- practice finding eigenvalues, eigenvectors, and transforming basis

Find eigenvectors $(M - \lambda_n I) v_n = 0$

$\lambda_1 = 1$ $\quad (M - \lambda_1 I) v_1 = \begin{pmatrix} -4-1 & 3 \\ -10 & 7-1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$

$\begin{pmatrix} -5 & 3 \\ -10 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \quad -5x + 3y = 0 \quad y = \frac{5}{3} x$

lower row gives same

pick $x = 3 \Rightarrow y = 5 \Rightarrow v_1 = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

$\lambda_2 = 2$ $\quad (M - \lambda_2 I) v_2 = \begin{pmatrix} -4-2 & 3 \\ -10 & 7-2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$

$\begin{pmatrix} -6 & 3 \\ -10 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \quad -6x + 3y = 0 \quad y = 2x$

pick $x = 1 \Rightarrow y = 2 \Rightarrow v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$M = \begin{pmatrix} -4 & 3 \\ -10 & 7 \end{pmatrix}$ Find eigenvalues.

find roots of characteristic polynomial

$P_M(\lambda) = \det(\lambda I - M) = 0$

do $\det(M - \lambda I) = \det\begin{pmatrix} -4-\lambda & 3 \\ -10 & 7-\lambda \end{pmatrix}$

$= (-4-\lambda)(7-\lambda) + 3 \cdot 10 = 0$

$= \lambda^2 - 7\lambda + 4\lambda - 28 + 30 = 0$

$= \lambda^2 - 3\lambda + 2 = 0$

factor $(\lambda - 2)(\lambda - 1) = 0$ $\lambda_1 = 1$ $\lambda_2 = 2$

or use quadratic equation $\lambda = \frac{3}{2} \pm \frac{1}{2}$

Change of basis matrix $P = \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix}$

$$P = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix} \qquad P^{-1} = \frac{1}{\det(P)} \operatorname{adj}(P) =$$

$$P^{-1} = \frac{1}{3 \cdot 2 - 1 \cdot 5} \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$$

$$D = P^{-1} M P = \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix} \begin{pmatrix} -4 & 3 \\ -10 & 7 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

$$u_B = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad u_V = P^{-1} u_B = \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -7 \end{pmatrix}$$

$$D u_V = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ -7 \end{pmatrix} = \begin{pmatrix} 3 \\ -14 \end{pmatrix}$$ in eigenvector basis

compare to $$P^{-1} M u_B = P^{-1} \begin{pmatrix} -4 & 3 \\ -10 & 7 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ -13 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix} \begin{pmatrix} -5 \\ -13 \end{pmatrix} = \begin{pmatrix} 3 \\ -14 \end{pmatrix} \checkmark$$

$D u_V = P^{-1} M u_B$ - calculations agree