1. A linear transformation from $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ has eigenvalues and vectors $\lambda_1 = 0.5, \xi_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\lambda_2 = 3, \xi_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

(a) Sketch, without any calculation, what happens to a vector in the first quadrant of the plane under this linear transformation.

(b) Calculate what happen to a vector $\begin{pmatrix} x \\ y \end{pmatrix}$ under this transformation.

2. Find the eigenvalues and eigenvectors of the linear operation specified by $T = \begin{pmatrix} -2 & -4 & 2 \\ -2 & 1 & 2 \\ 4 & 2 & 5 \end{pmatrix}$.

Note that this involves find the roots of a cubic equation. The best way to do that is to make an educated guess for one of the roots (try numbers like $-3, -2, ... 3$ since usually happens on exams). Once you find a root $r$, factor the polynomial into $(\lambda - r)(a\lambda b\lambda + c)$ and use the quadratic equation on the remaining quadratic.

3. We define a basis $\varepsilon$ using the basis vectors $\varepsilon_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\varepsilon_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ written in the standard basis. The linear transformation $L$ is $L_\varepsilon = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ when written in the $\varepsilon$ basis. What is $L$ when written in the standard basis?

4. The linear transformation $A$ has eigenvalue $\lambda$ and eigenvector $v$. Find one of the eigenvalue and eigenvector pairs for $A^2$ in terms of $\lambda$ and $v$.

5. Quantum Mechanics and other subjects make use of a quantity called the matrix exponential.

$$e^M = I + M + \frac{M^2}{2} + \frac{M^3}{6} + ...$$

If you were given a random n-by-n matrix, then this may be an unwieldy expression. Consider when $M$ is diagonalizable ($M = P^{-1}DP$).

(a) Write out the term for $M^2$ using the fact that it is diagonalizable and simplify.

(b) What does the $M^3$ term look like when you do the same simplification? How about the $M^n$ term?

(c) Write the matrix exponential in the form $e^M = Ae^DB$. What are $A$ and $B$? Why is this an easier calculation?