Cosmology

- Scale factor
- Cosmology à la Newton
- Cosmology à la Einstein
- Cosmological constant
- SN and dark energy
- Evolution of the Universe
Scale Factor

• Assume expansion of Universe is homogeneous and isotropic
• Then expansion can be described by a scale factor $a(t)$, such that

$$r(t) = a(t) \ r_0$$

where $r_0 = r$(now) and $a$ is dimensionless
Hubble Parameter

- Scale factor $a(t)$, such that $r(t) = a(t) \ r_0$
- Hubble law $v = Hr$
- Becomes

$$v = \frac{dr}{dt} = \dot{r} = \dot{a}r_0 = Hr = H a r_0$$

$$H = \frac{\ddot{a}}{a}$$
Cosmology à la Newton

- Model universe as homogeneous sphere with mass $M$ and radius $r$, consider test mass $m$ at surface. Then energy is:

$$E = km = K - U = \frac{1}{2} m v^2 - \frac{G M m}{r}$$

- Rewrite with scale factor

$$r = a r_0 \quad v = a \dot{r}_0 \quad M = \frac{4}{3} \pi r^3 \rho$$

$$\frac{1}{2} v^2 = \frac{1}{2} r_0^2 \dot{a}^2 = \frac{G M}{r} + k = G \frac{4}{3} \pi a^2 r_0^2 \rho + k$$
Cosmology à la Einstein

\[ \frac{1}{2} r_0^2 \dot{a}^2 = \frac{4}{3} \pi a^2 r_0^2 \rho + k \quad \Rightarrow \quad \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho + \frac{2k}{r_0^2 a^2} \]

\( k < 0 \): universe is bound, \( k > 0 \): universe is unbound

Change to relativistic version with parameters:

\( u = \) energy density

\( r_c = \) curvature of universe (always positive)

\( \kappa = \) curvature parameter +1=positive, 0=flat, -1=negative

\[ \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3c^2} u - \frac{\kappa c^2}{r_c^2 a^2} \]
Friedmann/Lemaître Equation

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3c^2} u - \frac{\kappa c^2}{r_c^2 a^2} + \frac{\Lambda}{3}
\]

Extra term with \( \Lambda = \text{“cosmological constant”} \) was added by Einstein.

Equivalent to adding a component to the Universe that has a constant energy density as a function of time, perhaps the energy of quantum fluctuations in a vacuum.

\[
u_\Lambda = \frac{c^2 \Lambda}{8\pi G}
\]
Energy densities

Rewrite Friedmann/Lemaitre equation in terms of energy densities.

\( u_r = \) radiation energy density

\( u_m = \) energy density of matter

\( u_\Lambda = \) energy density of cosmological constant or dark energy

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3c^2} \left( u_r + u_m + u_\Lambda \right) - \frac{\kappa c^2}{r_c^2 a^2}
\]
Evolution of energy densities

- Energy density of $\Lambda$ is constant in time.
- Energy density of matter (normal or dark)
  - Assume non-relativistic particles, then energy is dominated by rest mass
  - Rest mass is not red-shifted, so energy density varies like number density of particles, decreases as volume of universe increases

$$u_m(t) = n(t)\varepsilon = n(t)mc^2 = mc^2 \frac{N}{V} \propto a(t)^{-3}$$
Evolution of energy densities

- Energy density of radiation
  - Number density of photons as volume of universe increases
    \[ n(t) = \frac{N}{V} \propto a(t)^{-3} \]
  - Wavelength of photons increases as size of universe increases
    \[ \lambda(t) \propto a(t) \quad \text{so} \quad \varepsilon(t) = \frac{hc}{\lambda(t)} \propto a(t)^{-1} \]
  - Combine both factors
    \[ u_r(t) = n(t)\varepsilon \propto a(t)^{-3} a(t)^{-1} \propto a(t)^{-4} \]
Friedmann/Lemaître Equation

\[
\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2} \left( u_r + u_m + u_\Lambda \right) - \frac{\kappa c^2}{r_c^2 a^2}
\]

Previous equation

\[
\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2} \left( \frac{u_{r,0}}{a^4} + \frac{u_{m,0}}{a^3} + u_\Lambda \right)
\]

Know how \( u \)'s scale
Take \( \kappa = 0 \)

\[
\dot{a}^2 = H_0^2 \left( \frac{\Omega_{r,0}}{a^2} + \frac{\Omega_{m,0}}{a} + \Omega_\Lambda a^2 \right)
\]

\[
u_c = \frac{3H_0^2 c^2}{8\pi G} \quad \Omega_m = \frac{u_m}{u_c}
\]

\[
\dot{a} = H_0 \left[ \frac{\Omega_{r,0}}{a^2} + \frac{\Omega_{m,0}}{a} + \Omega_{\Lambda,0} a^2 \right]^{1/2}
\]
Energy densities

Critical density

\[ u_c = \rho_c c^2 = \frac{3H_0^2 c^2}{8\pi G} = 5200 \text{ MeV m}^{-3} \]

Express densities in terms of density parameters:

\[ \Omega_m = \frac{u_m}{u_c} , \ldots \]

From CMB curvature measurement:

\[ \Omega_r + \Omega_m + \Omega_\Lambda = 1.02 \pm 0.02 \]
Friedmann/Lemaître Equation

\[
\dot{a} = H_0 \left[ \frac{\Omega_{r,0}}{a^2} + \frac{\Omega_{m,0}}{a} + \Omega_{\Lambda,0} a^2 \right]^{1/2}
\]

\[
\ddot{a} = H_0^2 \left[ - \frac{\Omega_{r,0}}{a^3} - \frac{\Omega_{m,0}}{2a^2} + \Omega_{\Lambda,0} a \right]
\]

- Radiation and matter slow down expansion
- CC speeds up expansion
- Impossible to get static universe without CC
Matter slows down expansion

- Negatively curved space
- Flat space
- Positively curved space

\[ \Omega_0 < 1 \]
\[ \Omega_0 = 1 \]
\[ \Omega_0 > 1 \]
Einstein and Cosmology

- After Einstein wrote down the equations for General Relativity, he made a model of the Universe and found that the Universe had to be either expanding or contracting.
- He introduced a new term, the cosmological constant or $\Lambda$, in his equations representing a energy field which could create antigravity to allow a static model.
- After Hubble found the expansion of the Universe, Einstein called $\Lambda$ his greatest blunder.
- Quantum physics predicts some energy fields that act like $\Lambda$. 
Accelerating Universe
Accelerating Universe

• Hubble expansion appears to be accelerating

• Normal matter cannot cause acceleration, only deceleration of expansion

• Dark energy is required
  – may be cosmological constant
  – may be something else
  – major current problem in astronomy
Supernova constraints on $\Omega_s$

- Dashed vs solid are different SN samples
- Use curvature constraint $\Omega = 1.02 \pm 0.02$ to narrow range
Radiation Energy Density

Main component is CMB, star light is < 10%

\[ u_{\text{CMB}} = 0.260 \text{ MeV m}^{-3} \]

\[ \Omega_{\text{CMB}} = \frac{u_{\text{CMB}}}{u_c} = \frac{0.260 \text{ MeV m}^{-3}}{5200 \text{ MeV m}^{-3}} = 5.0 \times 10^{-5} \]

There are also likely neutrinos left over from the big bang, produced when nucleons froze out

\[ u_{\text{nu}} = 0.177 \text{ MeV m}^{-3} \]

\[ \Omega_{\text{CMB}} = \frac{u_{\text{CMB}}}{u_c} = \frac{0.177 \text{ MeV m}^{-3}}{5200 \text{ MeV m}^{-3}} = 3.4 \times 10^{-5} \]

Total for radiation: \( \Omega_{r,0} = 8.4 \times 10^{-5} \)
Matter Energy Density

• Matter in baryons (protons, neutrons, electrons): $\Omega_{\text{bary}} = 0.04$

• Matter in clusters (part dark): $\Omega_{\text{cluster}} = 0.2$

• Best estimate of all matter (baryons+dark): $\Omega_{m,0} = 0.3$

• Ratio of photons to baryons $\sim 2 \times 10^9$
## Consensus Model

<table>
<thead>
<tr>
<th>Component</th>
<th>$\Omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Photons</td>
<td>$5.0 \times 10^{-5}$</td>
</tr>
<tr>
<td>Neutrinos</td>
<td>$5.0 \times 10^{-5}$</td>
</tr>
<tr>
<td>Total radiation</td>
<td>$5.0 \times 10^{-5}$</td>
</tr>
<tr>
<td>Baryons</td>
<td>0.04</td>
</tr>
<tr>
<td>Dark matter</td>
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</tr>
<tr>
<td>Total matter</td>
<td>0.30</td>
</tr>
<tr>
<td>Cosmological constant</td>
<td>$\sim 0.7$</td>
</tr>
<tr>
<td>Curvature</td>
<td>1.02±0.02</td>
</tr>
</tbody>
</table>

- Hubble constant $= 70\pm5$ km s$^{-1}$ Mpc$^{-1}$
Energy density versus scale factor

\[ u_r \sim a^{-4} \]
\[ u_m \sim a^{-3} \]
\[ u_\Lambda = \text{const} \]

\[ z = 1/a - 1 \]

- Early times, \( z > 3600 \) or age < 47 kyr, were radiation dominated
- Matter dominated until 9.8 Gyr
- Current age 13.5 Gyr
• Different slopes of expansion in radiation vs matter dominated epochs
• Exponential expansion in $\Lambda$ dominated epoch (if like cosmological constant)
Proper distance versus redshift

- Proper distance reaches a limiting value of 14 Gpc
- Different distances are needed for different measurements: distance, angular size, luminosity
Review Questions

• As fractions of the critical density, what are the current energy densities of radiation, baryonic matter, dark matter, and dark energy?
• Derive the equation for the critical density
• How do radiation, matter, and the cosmological constant affect the rate of expansion of the Universe?
• When was the universe dominated by radiation, matter, and dark energy?