Lecture 36

Central force problems

\[ H = \frac{p_i^2}{2m_i} + \frac{p_j^3}{2m_j} + V(\mathbf{r}_i - \mathbf{r}_j) \]

0) change variables

\[ \bar{p} = \bar{p}_i + \bar{p}_j \]
\[ \bar{\mathbf{r}} = \frac{m_i \bar{\mathbf{r}}_i + m_j \bar{\mathbf{r}}_j}{m_i + m_j} \]
\[ \bar{M} = m_i + m_j \]
\[ \bar{\mathbf{r}} = \bar{\mathbf{r}}_i - m_i \frac{\bar{p}}{\bar{M}} = \frac{p_i m_j - p_j m_i}{m_i + m_j} \]
\[ \bar{r} = \bar{r}_i - \bar{r}_j \]

\[ \epsilon_i \epsilon_j \bar{r}_i \bar{r}_j = -i \hbar \delta_{ij} \]
\[ \epsilon_i \bar{r}_i \bar{r}_j = -i \hbar \delta_{ij} \]

All other commutators vanish

\[ H = \frac{\bar{p}^2}{2\bar{M}} + \frac{\bar{r}^2}{2\bar{M}} + V(\bar{r}) \]
\[ \bar{M} = \frac{m_i m_j}{m_i + m_j} \]
\[ \bar{\mathbf{L}} = \bar{\mathbf{r}} \times \bar{\mathbf{p}} \]

Schrödinger equation

\[ \left( -\frac{\hbar^2}{2\bar{M}} \nabla_r^2 - \frac{\hbar^2}{2\bar{M}} \nabla_i^2 + V(\bar{r}) - \bar{M} \right) \psi(\bar{r}, \bar{p}) = 0 \]

\[ \nabla_r^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} - \frac{\bar{r}^2}{r^2 \bar{M}^2} \]

Try a solution of the form

\[ \psi(\bar{r}, \bar{p}) = \frac{1}{(2\pi \hbar)^{3/2}} e^{i \bar{p} \cdot \bar{r}/\hbar} R_e(r) Y_n(\theta \phi) \]
\[-\frac{\hbar^2}{2M} \nabla^2 \psi(R, \vec{r}) = \frac{p^2}{2M} \psi(R, \vec{r})\]

\[\frac{\hbar^2}{r^2} \psi(R, \vec{r}) = \ell(\ell+1) \psi(R, \vec{r})\]

Putting everything together

\[\left( \frac{p^2}{2M} - \frac{\hbar^2}{2M} \frac{d^2}{dr^2} - \frac{\hbar^2}{2M} \frac{2 \frac{d}{dr}}{r} + \frac{\hbar^2 \ell(\ell+1)}{r^2} + V(r) - E \right) \frac{1}{(2\pi\hbar)^{3/2}} \psi(R, \vec{r}) \cdot \frac{i \sigma \cdot \vec{P}}{\hbar^2} \times y_\ell^m(\varphi) = 0\]

We can cancel \(\frac{1}{(2\pi\hbar)^{3/2}} \psi(R, \vec{r}) \cdot \frac{i \sigma \cdot \vec{P}}{\hbar^2} \times y_\ell^m(\varphi)\) to get a differential equation in \(R_0(r)\)

\[\left( \frac{d^2}{dr^2} + \frac{2 \frac{d}{dr}}{r} - \frac{\hbar^2 \ell(\ell+1)}{r^2} - \frac{2M}{\hbar^2} \left( \frac{p^2}{2M} - E + V(r) \right) \right) R_0(r) = 0\]

Defining \(E = E - \frac{p^2}{2M} = \text{rest energy}\)

\(\text{3 write } \ R_0(r) = \frac{U_\ell(r)}{r}\)

For a bound state this should vanish at \(\infty\) and be well behaved near the origin

\(\Rightarrow \ U_\ell(\infty) = U_\ell(0) = 0\)

Using this in the equation we get an equation that looks like the one-dimensional Schrödinger equation.
\[
\left( \frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} - \frac{2\mu}{\hbar^2} \left( -\varepsilon + V(r) \right) \right) U_e(r) = 0
\]

for bound states we expect that the rest energy eigenvalue is negative:

\[
\frac{2\mu}{\hbar^2} \varepsilon = -b^2
\]

special case: hydrogen-like atom

\[
V(r) = -\frac{Ze^2}{r}
\]

define

\[
a = \frac{2\mu Ze^2}{\hbar^2}
\]

\[
\left( \frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} - b^2 + \frac{a}{r} \right) U_e(r) = 0
\]

for equations with low order poles the solution is analytic except possible at the poles:

\[
U_e(r) = r^\gamma \sum_{n=0}^{\infty} C_n r^n
\]
the solution method is simplified if we build in the correct asymptotic boundary condition

4. large $r$

\[ \left( \frac{d^2}{dr^2} - b^2 \right) u_e(r) = 0 \quad u_e(r) \sim e^{\pm br} \]

for the correct boundary condition $u \to e^{-br}$

5. small $r$

\[ \left( \frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} \right) u_e(r) = 0 \quad u \to r^\beta \]

$\beta(\beta-1) = \ell(\ell+1)$

\[ \beta^2 - \beta - \ell(\ell+1) = 0 \]

This is a quadratic equation in $\beta$

\[
\beta = \frac{1}{2} \left( 1 \pm \sqrt{1+4\ell(\ell+1)} \right) \\
= \frac{1}{2} \left( 1 \pm \sqrt{(2\ell+1)^2} \right) \\
= \frac{1}{2} \left( 1 \pm 2\ell+1 \right) = \begin{cases} \ell, & \text{does not vanish at 0} \\ \ell+1, & \end{cases}
\]

$\beta = \ell+1$

Putting these together we assume a solution of the form

\[ u_e(r) = e^{-br} \sum_{n=0}^{\infty} c_n r^{\ell+\alpha} \]
\[
\frac{d}{dr} \left( e^{-br} r^{l+1+n+\alpha} \right) =
\]
\[
e^{-br} \left( -br^{l+1+n+\alpha} + (l+1+n+\alpha) r^{l+n+\alpha} \right)
\]
\[
\frac{d^2}{dr^2} \left( e^{-br} r^{l+1+n+\alpha} \right) =
\]
\[
e^{-br} \left( b^2 r^{l+1+n+\alpha} - 2b (l+1+n+\alpha) r^{l+n+\alpha} + (l+1+n+\alpha)(l+n+\alpha) r^{l+n+\alpha-1} \right)
\]
\[
\left( \frac{d^2}{dr^2} - \frac{e(e+1)}{r^2} - b^2 + \frac{\alpha}{r} \right) e^{-br} \sum_{n=0}^{\infty} c_n r^{\alpha+n+1} = 0
\]
\[
e^{-br} \sum_{n=0}^{\infty} c_n \left( b^2 r^{l+1+n+\alpha} - 2b (l+1+n+\alpha) r^{l+n+\alpha} + (l+1+n+\alpha)(l+n+\alpha) r^{l+n+\alpha-1} - e(e+1) r^{l+n+\alpha-1} - b^2 r^{l+1+n+\alpha} + \alpha r^{l+n+\alpha} \right)
\]

Now we cancel out the \( b^2 \) and \( e(e+1) \) terms. What remains

\[
e^{-br} \sum_{n=0}^{\infty} c_n \left( -2b(l+1+n+\alpha) + \alpha \right) r^{l+n+\alpha}
\]
\[
\left( (l+\alpha)(l+\alpha+2e+1) r^{l+n+\alpha-1} \right) = 0
\]

We equate the coefficients of common powers of \( r \) to 0.
\( l + d - 1 \)

\( a(a+2l+1) c_0 = 0 \)

To get a non-zero \( c_0 \) we must choose \( d = 0 \) or \( e = -2l - 1 \)

For \( d = 0 \) \( u \rightarrow r^{l+1} \) at the origin

For \( d = -2l - 1 \) \( u \rightarrow r^{-l} \) at the origin

The choice \( d = -2l - 1 \) gives the wrong behavior at the origin.

i.e. we choose \( d = 0 \) as this allows for a non-zero value of \( c_0 \) and leads to a well-behaved wave function at \( r = 0 \)

With this choice

\[
0 = \sum_{n=0}^{\infty} \frac{1}{c_n} \left[ (-2b(n+2l+1) + a) r^{l+n+1} + n(n+2l+1) r^{l+n-1} \right] = 0
\]

Let \( n = m - 1 \) in the first term \( = n - 1 \)

\[
\left[ c_{n-1} (a - 2b(n+2l)) + c_n (n)(n+2l+1) \right] r^{l+n+1} = 0
\]

This gives the recursion

\[
c_n = - \frac{(a-2b(n+2l))}{n(n+2l+1)} c_{n-1}
\]

\[
c_{n+1} = - \frac{(a-2b(n+2l+1))}{(n+1)(n+2l+2)} c_n
\]
given \( C_0 \), this gives all of the other coefficients

for large \( n \)

\[
\frac{C_{n+1}}{C_n} = -\frac{-2bn}{n^2} = \frac{2b}{n}
\]

this is the same behavior as

\[
e^{2br} = \sum_{n=0}^{\infty} \frac{(2br)^n}{n!}
\]

\[
\frac{C_{n+1}}{C_n} = \frac{(2b)^{n+1}}{(n+1)!} \frac{n}{(2b)^n} = \frac{2b}{n+1}
\]

if we multiply \( e^{2br} \times e^{-br} \) we see that we generate a solution that violates the boundary condition at \( \infty \).

this will not happen if the series terminates

setting \( C_{n+1} = 0 \Rightarrow \)

\[
a - 2b(n+e+1) = 0
\]

in this case the highest power is \( n \) and

\[
b = \frac{a}{2(n+e+1)}
\]

\[
b^2 = \frac{a^2}{4(n+e+1)^2}
\]
\[ \epsilon = - \frac{\hbar^2}{2m} b^2 = \]
\[ = - \frac{\hbar^2}{2m} \frac{a^2}{4(n+\ell+1)^2} \]
\[ = - \frac{\hbar^2}{2m} \frac{1}{4(n+\ell+1)^2} \cdot \frac{4m^2Ze^2}{\hbar^2} \]
\[ = - \frac{mZe^2}{2\hbar^2} \frac{1}{(n+\ell+1)^2} \]

These are the internal energy eigenvalues for this problem. The corresponding wave functions are

\[ \psi_n(z) = r^{\ell+1} e^{-\frac{a}{2r}} \sum_{m=0}^{\infty} \frac{C_m r^m}{(n+\ell+1)} \]

We can also find the structure of the polynomial

\[ C_1 = - \frac{2b(n+\ell+1) - 2b(\ell+1)}{1(2\ell+1)} = - \frac{2b}{1(2\ell+1)} C_0 \]
\[ C_2 = - \frac{2b(n-1)}{2(2\ell+3)} C_1 = - \frac{2b}{2(2\ell+3)} \frac{(2b)^2(n)(n-1)}{1\cdot 2\cdot (2\ell+2)(2\ell+3)} C_0 \]
\[ C_k = \left( \frac{n(n-1) \cdots (n-k+1)}{(2\ell+2)(2\ell+3) \cdots (2\ell+k)} \right) C_k = \left( \frac{(2b)^k}{(1\cdot 2 \cdots k)(2\ell+2)(2\ell+3) \cdots (2\ell+k)} \right) C_k = \]
we can write this as:

\[ C_k = (-1)^k (2b)^k \frac{n! \Gamma(2k+1)}{(n-k)! k! (2k+K!)!} \]

\[ \sum_{k=0}^{n} (-1)^k \frac{n!}{(n-k)!} \frac{1}{k!} \frac{(2k+1)!}{(2k+K+1)!} (2br)^k \]

the factors \( n! (2k+1)! \) are independent of \( k \)
and can be absorbed in the coefficient
so which is determined by normalization.

Note

\[ L_{\alpha}^n(x) = \sum \frac{(n+1)!}{(n-k)! k! (\alpha+k)!} (-1)^k x^k \]

if we set \( \alpha = 2k+1 \)

\[ L_{n}^{2k+1}(2br) = \sum_{k=0}^{n} \frac{(n+2k+1)!}{(n-k)! k! (2k+1+k)!} (2br)^k \]

this differs from the above polynomials
by factors that are independent of \( k \)

\[ U_k(r) = N E^{-br} (2br)^{r+1} L_{n}^{2r+1}(2br) \]

\[ R_k(r) = N E^{-br} (2br)^r L_{n}^{2r+1}(2br) \]
to calculate the normalization integral

\[ I = \int_0^\infty R_0(r) r^2 dr = \int_0^\infty N e^{\frac{-2br}{(2br)^{2+1}}} \left(\frac{1}{2b}\right)^2 L_n\left(\frac{2b}{2br}\right)^2 dr \]

let \( V = 2br \) \( dv = 2b \, dr \)

\[ \frac{N}{(2b)^2} \int_0^\infty e^{-V} V^{2+2} L_n(V)^2 \, dV = 1 \]

this can be computed using properties of \( L_\alpha^n(x) \)

\[ L_\alpha^{n+1}(n+1) = (2n+\alpha+1 - \alpha x) L_\alpha^n - (n+\alpha) L_{\alpha}^{n-1} \]

\[ \int L_\alpha^n(x) e^{-x} x^{\alpha} \, dx = \frac{n!(n+\alpha)}{\alpha!} \]

\[ \int_0^\infty L_\alpha^n(V) L_{\alpha}^{n+1}(V) V^{2\alpha+1} e^{-V} \, dV = 0 = \]

\[ (2n+2\alpha+2) \int_0^\infty L_\alpha^n(V)^2 V^{2\alpha+2} e^{-V} \]

\[- (2\alpha+1) \int_0^\infty L_\alpha^n(V)^2 V^{2\alpha+2} e^{-V} \]

\[ = (2\alpha+1) \int_0^\infty L_\alpha^n(V) L_{\alpha}^{n+1}(V) V^{2\alpha+1} e^{-V} \, dV \]

\[ \int_0^\infty L_\alpha^n(V)^2 V^{2\alpha+2} e^{-V} \, dV = \frac{2n+2\alpha+2 (2\alpha+n+1)!}{2\alpha+1 \cdot n!} \]
using this we can read off the normalization

\[ N^2 = (2b)^3 \left( \frac{2n+\ell+2}{2\ell+1} \frac{(2\ell+n+1)!}{n!} \right)^{-1} \]

\[ N' = (2b)^{3/2} \sqrt{\frac{2\ell+1}{2n+2(2\ell+n+1)!}} \]

\[ \psi = \frac{i}{(2\pi\hbar)^{3/2}} e^{-\frac{\mathbf{\mathbf{R}}^2}{\hbar}} \psi_m^{\ell} (\mathbf{\mathbf{q}}) \sqrt{\frac{2\ell+1}{2n+2(2\ell+n+1)!}} (2b)^{3/2} \]
\[ \times e^{-\frac{b\mathbf{r}}{r}} (2br)^{\ell} L_n^{2\ell+1} \left( 2br \right) \]