Lecture 5

change of basis

\[ U|a_n\rangle = |b_n\rangle \]

\[ U = \sum_{n=1}^{N} |b_n\rangle \langle a_n| \]

\[ U^\dagger = \sum_{n=1}^{N} |a_n\rangle \langle b_n| \]

\[ UU^\dagger = U^\dagger U = I \]

Operators satisfying this condition are called unitary operators. We note

(i) any change of basis is implemented by a unitary operator

(ii) any unitary operator defines a change of basis

Eigenvalue problem for unitary operators

Assume

\[ U|u_n\rangle = |u_n\rangle \]

\[ \langle u_n | u_m \rangle = \langle u_n | U^\dagger U | u_m \rangle = \langle u_n | U^\dagger | u_m \rangle \langle u_m | U | u_n \rangle = \delta_{mn} \]

\[ \langle u_n | U^\dagger U | u_m \rangle = \langle u_n | u_m \rangle U^\dagger U | u_m \rangle = \langle u_n | u_m \rangle U^\dagger \delta_{mn} = \delta_{mn} \langle u_n | u_m \rangle \]
\[ 0 = \langle u_n | u_m \rangle (1 - u_n^* u_m) \]

if \( m = n \)

\[ 0 = \langle u_n | u_n \rangle (1 - u_n^* u_n) \]

\[ u_n^* u_n = 1 \quad \Rightarrow \quad u_n = e \]

if \( \langle u_n \rangle \neq 0 \) then \( u_n^* u_n = 1 \) \( \Rightarrow \) \( u_n = e \)

if \( m \neq n \)

\[ 0 = \langle u_n | u_m \rangle (1 - u_n^* u_m) \]

multiply by \( u_n \)

\[ 0 = \langle u_n | u_m \rangle (u_n - u_n^* u_m) \]

\[ = \langle u_n | u_m \rangle (u_n - u_m) \]

thus if \( u_n \neq u_m \) we must have \( \langle u_n | u_m \rangle = 0 \)

This shows that the eigenvalues of a unitary operator are on the unit circle and if the eigenvalues are different then the eigenvectors are orthogonal.

In general

\[ \det (\langle k_n | u_n \rangle - \lambda \delta_{mn}) = 0 \]

has \( n \) roots, so there are \( n \) eigenvalues

\[ \langle u_n \rangle = \prod_{m=1}^{m+n} \left( \frac{2 u_n^* - u_n}{u_n - u_m} \right) \epsilon \quad \langle u_n \rangle = \frac{\langle u_n \rangle}{\langle v_n | u_n \rangle \frac{1}{2}} \]
What is the difference between Hermitian and unitary operators:

\[ A = A^+ \]

\[ A = \sum \alpha_n |a_n \rangle \langle a_n| \]

replace \( a_n \) by \( e^{i\alpha_n} \)

\[ e^{iA} = \sum \alpha_n |e^{i\alpha_n}a_n \rangle \langle a_n| \]

These operators both have the same eigenvectors – \( e^{iA} \) is a complex function of \( A \).

For the spin \( S_z \), \( e^{iS_z/\hbar\Theta} \) both have the eigenvector \( 1 \rightarrow z \leftrightarrow -z \).

So unitary operators can behave like observables with complex values.

A second example is

\[ U = \frac{1 - iA}{1 + iA} \quad A = i \frac{2u - 1}{u + 1} \]

These are called Cayley transformations.
Complementary observables

consider a beam of particles
with spin up in the x direction
measure $S_z = \frac{1}{2}$ of the particles
half have spin up in the z direction,
half have spin down in the z direction.

next consider a beam with
spin down in the x direction
measure $S_z = \frac{1}{2}$ result is exactly
the same.

by measuring $S_z$ we have no
way of knowing whether the
initial beam was $|+\rangle_x$ \quad $|\rangle_x$,

so $S_z$ $S_x$ are complementary observables

General Definition $A, B$ complementary

$|\langle a_n | b_m \rangle|^2 = |\langle a_n | b_m \rangle \langle b_m | a_n \rangle| = \frac{1}{N}$.

all probabilities are identical
Now we want to show how to represent and operate as a polynomial in 2 complementary observables.

1. Start with $A$
   
   $A|\alpha_n\rangle = \alpha_n|\alpha_n\rangle$ \hspace{1cm} n = 1 \ldots N

2. Define a unitary change of basis operator by
   
   $U|\alpha_n\rangle = |\alpha_{n+1}\rangle$ \hspace{1cm} n ≠ N
   
   $U|\alpha_n\rangle = |\alpha_n\rangle$

   with this definition we have
   
   $U^N|\alpha_n\rangle = |\alpha_n\rangle$ \hspace{1cm} or
   
   $(U^N - I)|\alpha_n\rangle = 0$

   $(U^N - I)\sum l_{\alpha_n}\langle \alpha_n | = (U^N - I)I = (U^N - I) = 0$.

   Let $|\alpha_n\rangle$ be an eigenstate of $U$. Then
   
   $(U^N - I)|\alpha_n\rangle = (U^N - 1)|\alpha_n\rangle = 0$

   $U^N - 1$ the roots of this polynomial equation are $\alpha_n = \exp^{2\pi i n/N}$
eigenvalues $u_n = e^{\frac{2\pi i n}{P}}$ $u_n^N = 1$

**Next note**

$$\left(\frac{u}{u_n}\right)^N - 1 = 0$$

$$x^N - 1 = (x - 1)(1 + x + x^2 + \cdots + x^{N-1})$$

If $x = \frac{u}{u_n}$

$$\left(\frac{u}{u_n}\right)^N - 1 \equiv \frac{1}{u_n} (u - u_n) (1 + \frac{u}{u_n} + \left(\frac{u}{u_n}\right)^2 + \cdots + \left(\frac{u}{u_n}\right)^{N-1})$$

$$u^N - 1 \equiv (u - u_1) (u - u_2) \cdots (u - u_N)$$

We expect that up to normalization

$$\left(\frac{u}{u_n}\right) \sim |u_n| < |u_n|$$

To show this consider

$$\sum_{m=0}^{N-1} \left(\frac{u}{u_n}\right)^m = \sum_{m=0}^{N-1} \left(\frac{u}{u_n}\right)^m \sum_{m=0}^{N-1} \left(\frac{u}{u_n}\right)^m = \sum_{m=0}^{N-1} \sum_{k=1}^{N} (\frac{u_k}{u_n})^m |u_k| < |u_k| =$$

$$\sum_{k=1}^{N} \left(\frac{u_k}{u_n}\right)^N - 1$$

Consider 2 cases

1. $k = n$ then $\sum_{m=0}^{N-1} \left(\frac{u_k}{u_n}\right)^m = \sum_{m=0}^{N-1} 1^m = N$

2. $k \neq n$ then

$$e^{\frac{2\pi i (k-n)N}{P}} \frac{2\pi i (k-n)}{P} - 1 = 0$$
Combining these two results we get
\[ |u_n\rangle \langle u_n| = \sum_{m=0}^{\infty} \left( \frac{u_n}{u_m} \right)^m = \sum_{m=1}^{\infty} \left( \frac{u_m}{u_n} \right)^m \]
(we used \( 1 = \left( \frac{u_n}{u_n} \right)^m \))

Now we show that \( u \) is a complementary observable to \( A \)

\[ \langle a_K \mid u_n \rangle^2 = \langle a_K \mid u_n \rangle \langle u_n \mid a_K \rangle = \]
\[ = \langle a_K \mid \sum_{m=0}^{\infty} \left( \frac{u_n}{u_m} \right)^m a_K \rangle = \]
\[ = \frac{1}{N} \sum_{m=0}^{\infty} \langle a_K \mid a_{K+m} \rangle \left( \frac{u_n}{u_m} \right)^m \]
\[ = \frac{1}{N} \left( \frac{u_n}{u_m} \right)^m = \frac{1}{N} \frac{u_n}{u_m} = \frac{1}{N} \]
\[ \text{if } m = 0 \]

This shows that these are complementary.

- If we fix a state of \( A \) and then measure \( u \), we lose all information about the original state of \( A \).

Just like in the case of \( \text{Sx S} \), \( \langle a_m | b_m \rangle \) is not measurable — we have some freedom in choosing phases.
we make the phase convention

\[ \langle u_n | a_n \rangle = \frac{1}{\sqrt{n}} \]

From this we get all of the others:

\[ \langle a_R | u_n \rangle \langle u_n | a_n \rangle = \]

\[ \frac{1}{N} \langle a_R | \sum_{m=1}^{N} \frac{1}{u_n} | a_n \rangle = \]

\[ \frac{1}{N} \left( \frac{1}{u_n} \right)^k = \langle a_R | u_n \rangle \frac{1}{\sqrt{N}} \]

\[ \therefore \quad \langle a_R | u_n \rangle = \frac{1}{\sqrt{N}} e^{-\frac{2\pi i n k}{N}} \]

Taking conjugates

\[ \langle u_n | a_R \rangle = \frac{1}{\sqrt{N}} e^{\frac{2\pi i n k}{N}} \]

Next we introduce another unitary operator \( V \) that shifts the \( |u_n\rangle \) basis:

\[ \langle u_n | V = \langle u_{n+1} | \quad n \neq 0 \]

\[ \langle u_n | V = \langle u_n | \]
Following the same steps that we used with $u$

(1) $V^n \cdot 1 = 0$

(2) $V |u_n \rangle = u_n |u_n \rangle \quad u_n = 1 \quad u_n = e^{\frac{i \pi n}{m}}$

(3) $|u_n \rangle \langle u_n | = \frac{1}{m} \sum_{m=0}^{m-1} \left( \frac{V}{u_n} \right)^m = \frac{1}{m} \sum_{m=0}^{m-1} \left( \frac{V}{u_n} \right)^m$

Now we show that $u$ and $v$ are complementary operators

$\langle u_k | v_n \rangle \langle v_m | u_k \rangle =$

$\frac{1}{N} \left( \sum_{m=0}^{m-1} \left( \frac{V}{u_n} \right)^m \right) \langle u_k | u_k \rangle =$

$\frac{1}{N} \sum_{m=0}^{m-1} \left( \frac{V}{u_n} \right)^m \langle u_{R+m} | u_k \rangle = \frac{1}{N} \frac{1}{m} = \frac{1}{N}$

$\langle u_R | v_n \rangle^2 = \frac{1}{N}$

so $u$ and $v$ are complementary.

Next we compute $\langle u_k | u_m \rangle$ — as before we are free to choose the phase — we choose

$\langle u_n | u_m \rangle = \frac{1}{\sqrt{N}}$. 
To calculate the other phases note
\[ \langle u_m | u_m \rangle \langle u_m | u_n \rangle = \frac{1}{V^2} \langle u_m | u_n \rangle \]
\[ \frac{1}{N} \left\langle u_m \left| \sum_{n=1}^{N} \left( \frac{v}{u_m} \right)^n \right| u_n \right\rangle = \]
\[ \frac{1}{N} \sum_{n=1}^{N} \langle u_m | u_n \rangle \frac{1}{(u_m)^n} = \frac{1}{N} \left( \frac{1}{u_m} \right)^R \]
\[ \Rightarrow \langle u_m | u_n \rangle = \frac{1}{\sqrt{N}} e^{-\frac{-2\pi i m n}{R}} \]
and
\[ \langle u_m | u_m \rangle = \frac{1}{\sqrt{N}} e^{\frac{2\pi i m n}{R}} \]

With these phase conventions,
\[ |a_m \rangle = \sum |u_n \rangle \langle u_n | a_m \rangle \]
\[ \frac{1}{\sqrt{N}} e^{-\frac{-2\pi i m n}{R}} \]
\[ |u_m \rangle = \sum |u_n \rangle \langle u_n | u_m \rangle \]
\[ \frac{1}{\sqrt{N}} e^{\frac{2\pi i m n}{R}} \]

From which we see
\[ (u_m \rangle = (a_m \rangle \]
This means that $V$ is a unitary function of $A$

what have we gained from this

(1) recall that every $S(A)$ can be expressed as a polynomial in $A$; but this does not exhaust all possible operators.

(2) consider a general operator $V$

$O = \| \sum_{u_n} \langle u_n | 0 | u_m \rangle \langle u_m | \rangle$

since $\langle u_m | = \langle u_m | (V)$

$O = \sum_{m} \frac{2}{m^{\circ}} \langle u_n | 0 | u_m \rangle \langle u_m | (V)\langle u_m | X | u_m \rangle \langle u_m | (V)^{(m-n)^{\circ}}\rangle$

$\frac{1}{N} \sum_{u_n} (\frac{\langle u_m | (V)}{N})^{(m-n)}$

This shows that any operator can be expressed as a polynomial of degree $N-1$ in both $U$ and $V$.\]
we only need a pair of complementary observables.

Note the $UV \neq VU$ - to see this:

$$UV = 2 \langle u_n | u_n \rangle$$

$$= 2 \langle u_n | u_n \rangle$$

$$VU = 2 \langle u_{n-1} | u_{n-1} \rangle$$

Note $V|u_n\rangle = 2 \langle u_m | u_m \rangle V|u_n\rangle = 2 \langle u_m | u_{m+1} | u_n \rangle = |u_{n-1}\rangle$

Let $n' = n+1$ in $UV$

$$UV = 2 \langle u_{n-1} | u_{n-1} \rangle$$

$$= 2 \langle u_{n-1} | u_{n-1} \rangle e^{-2\pi i/n}$$
we see that $\mathbf{E}_{n} \neq 0$, however we can also write the polynomial with $V$'s on the left and $U$'s on the right

$$0 = \mathbf{E}_{n} < u_{1} | 0 | u_{m} > < u_{m} |$$

$$V | u_{n} > = | u_{n-k} > \quad n-k = m \pmod{n}$$

$$k = n-m$$

$$0 = \sum_{k=0}^{n-m} < u_{n-k} | 0 | u_{m} > \frac{1}{\lambda} \left( \frac{U}{U_{m}} \right)^{k}$$

$$= \sum_{k=0}^{n-m} \frac{< u_{n-k} | 0 | u_{m} >}{U_{m}} U_{m}^{k}$$

where the powers of $U_{m}U$ can be reduced to numbers between 0, $n-1$ using $V^{n} = U_{m}^{n} = I$ and $V^{-n} = U_{m}^{-n} = I$

we can write the expression in the form

$$0 = \sum_{m,n}^{m} u_{n} v_{m}$$

$$= \sum_{m,n}^{m} v_{m} u_{n}$$
while it is possible to express general operators in terms of different pairs of operators that not necessarily complementary - the \( u \) and \( v \) that we have constructed standard basic building block in quantum systems.

The \( u \) and \( v \) are called an irreducible set of operators. For homework you will show that any operator that commuted with both of these operators is necessarily a multiple of the identity (this is what we mean by irreducible).

Example

\[
B = b_1 \langle u_1 | u_2 \rangle + b_2 | u_3 \rangle \langle u_2 | u_3 \rangle
\]

we note \( \langle u_1 | u_2 \rangle = \langle u_1 | u_1 \rangle V = \frac{2}{3} \frac{U^k}{e^{2\pi ik/3}} V \),

\( \langle u_2 | u_1 \rangle = \langle u_2 | u_2 \rangle V^2 = \frac{1}{3} \frac{U^k}{e^{2\pi ik/3}} V^2 \)

\[
= \frac{b_3}{3} (1 + U + U^2) + \frac{b_1}{3} (V + e^{-\frac{2\pi i}{3}} U + e^{\frac{2\pi i}{3}} U^2) + \frac{b_1}{3} (V^2 + e^{-\frac{2\pi i}{3}} U V + e^{\frac{2\pi i}{3}} U^2 V^2)
\]
This shows clearly that the $B$ can be expressed as a degree 2 polynomial in the $U$ and $V$ operators.

Commuting observables.

So we used measurements of a single quantity to construct the vector space in the quantum theory.

The number of outcomes determines the dimensionality of the vector space.

In many cases the quantum system may have additional degrees of freedom that are not affected by the measurement.

In an electron it has a linear momentum and an intrinsic spin.

With a little work we can devise a Stern-Gerlach-like device that produces a silver atom with
a given value of momentum and $z$ component of spin, we can devise experiments that measure one without disturbing the other.

Momentum and spin are examples of commuting observables. $A$ and $B$ are commuting observables if $AB = BA$.

We write this as $[A, B] = 0$.

If $|a_n\rangle$ is an eigenvector of $A$ with eigenvalue $a_n$, then

$AB|a_n\rangle = BA|a_n\rangle = a_nB|a_n\rangle$ which means $|a_n\rangle$ is also an eigenvector of $A$ with eigenvalue $a_n$.

In this sense we see the measurement of $B$ has not disturbed to result of a measurement of $A$ — we can measure $A$ before or after we measure $B$ and get the same result.
Consider

\[ \langle a_n \rangle = P(A) \{ \text{polynomials} \} \]
\[ \langle b_m \rangle = Q(B) \]

Since \( AB = BA \) \( \{ P(A)Q(B) = Q(B)P(A) \} \)

\[ A^k B^k = B^k A^k \]

\[ P(A)Q(B) = \langle a_n \rangle \langle b_m \rangle \]
\[ = \langle b_m \rangle \langle a_n \rangle \]

Clearly

\[ B \langle b_m \rangle \langle a_n \rangle = b_m \langle b_m \rangle \langle a_n \rangle \]
\[ A \langle b_m \rangle \langle a_n \rangle = a_n \langle b_m \rangle \langle b_m \rangle \]

(it is possible for some of these products to vanish)

However

\[ I = (\sum \langle a_n \rangle) (\sum \langle b_m \rangle) \]

means any vector has an expansion of the form

\[ \langle c \rangle = \sum_{mn} \langle a_n \rangle \langle b_m \rangle \ c_{mn} \]
In this case the possible states of the system are labeled by 2 numbers a\text{ and } b\text{.} To fix this state we must measure A and B in any order and select the \(n^{th}\) and \(m^{th}\) outcomes of A resp. B.

Example — assume that the system consists of 2 silver atoms — each can be in different spin states

\[ A = S_z^1 \quad \text{z component of spin of atom 1} \]

\[ B = S_y^2 \quad \text{y component of spin of atom 2} \]

Since we can do independent measurements of \(S_z\) on particle 1 and \(S_y\) on particle 2 — a basis of states for this system is

\[ |\uparrow_z \uparrow_y> |\uparrow_z \downarrow_y> |\downarrow_z \uparrow_y> |\downarrow_z \downarrow_y> \]

because these involve different silver atoms \(S_z^1\) and \(S_y^2\) are \underline{not} complementary. They satisfy \([S_z^1 S_y^2] = 0\)
In general we may need to measure the values of several commuting observables to uniquely specify the state of the system.

\[ A_i, A_k \quad [A_i, A_k] = 0 \quad A_i = A_i^f \]

They are a complete set of commuting observables measuring the values of all of them gives a complete specification of the state of the system (this depends somewhat on what we call a system).

Then the simultaneous eigenstates are

\[ |\alpha_{in}, a_{k\ell n} \rangle \]

\[ A_k |\alpha_{in}, a_{k\ell n} \rangle = \alpha_{k\ell n} |\alpha_{in}, a_{k\ell n} \rangle \]