Lecture 6
complementary observables
A |an>  B |bn>  n=1...N

|<an|bm>|^2 = \frac{1}{N} \Rightarrow A, B complementary

Start with A |an>  A|an> = \lambda_n |an>

Define unitary change of basis by

U|an> = |an+1>  n=1
U|an> = |an>

Then

U^n |an> = |an>

U^n = U^n \sum |an><an| = \sum |an><an| = I

eigenvalue problem

U|\Psi> = \lambda|\Psi>

(\lambda^{N-1}) = 0 characteristic polynomials

\lambda^n = e^{\frac{2\pi i n}{N}} = \omega^n  n=1...N  \omega = 0...N-1
then we found
\[ \sum_{n=0}^{N-1} \left( \frac{u}{u_m} \right)^n |u_m\rangle = N |u_m\rangle \]
\[ \sum_{n=0}^{N-1} \left( \frac{u}{u_m} \right)^n |u_k\rangle = 0 \quad k \neq m \]

this led to
\[ |u_n\rangle \langle u_n| = \frac{1}{N} \sum_{m=0}^{N-1} \left( \frac{u}{u_m} \right)^m \]
\[ \langle a_k | u_n \rangle \langle u_n | a_k \rangle = \frac{1}{N} \sum_{m=0}^{N-1} \langle a_k | a_{k+1} \rangle u_m = \frac{1}{N} \]

this shows \( A \) and \( U \) are complementary operators.

we are free to choose phases
\[ \langle u_n | a_{k+1} \rangle = \frac{1}{\sqrt{N}} \]

(like choosing \( 1^+ \rangle_x = \frac{1}{\sqrt{2}} 1^+ \rangle_z + \frac{1}{\sqrt{2}} 1^- \rangle_z \)
\[ \langle a_k | u_n \rangle \langle u_n | a_k \rangle = \frac{1}{N} \langle a_k | u_n \rangle \]
\[ \frac{1}{N} \sum_{k=0}^{N-1} \langle a_k | (u/u_n) \rangle \langle u_n | a_k \rangle = \frac{1}{N} \sum_{k=0}^{N-1} \langle a_k | a_k \rangle = \frac{1}{N} \sum_{k=0}^{N-1} \langle u_n | u_n \rangle = \frac{1}{N} \]
\[ \frac{1}{N} e^{-2\pi i m \rho} \]
\[ \langle a_e | u_n \rangle = \frac{1}{\sqrt{N}} e^{-\frac{2\pi i n}{\beta}} \]

Next we define a new change of basis vector

\[ \langle u_n | V = \langle u_{n+1} | \]

\[ \langle u_n | V = \langle u_{n} | \]

as with \( u \) we find

\[ \langle u_n | V^n = \langle u_n | \]

\[ \sum_{n=0}^{N-1} \langle u_n | V^n (V^n-1) = V^n - 1 = 0 \]

eigenvalues \quad \nu_n = u_n = e^{\frac{2\pi i n}{\beta}}

\[ \langle u_n | u_n \rangle = \frac{1}{N} \sum_{m=0}^{N-1} \left( \frac{\nu}{\nu} \right)^m = \frac{1}{N} \sum_{m=1}^{N} \left( \frac{\nu}{\nu} \right)^m \]

\[ \langle u_e | V^n \rangle \langle u_n | u_e \rangle = \]

\[ \frac{1}{N} \sum_{m=0}^{N-1} \langle u_e | \left( \frac{\nu}{\nu} \right)^m \rangle \langle u_e | u_{n+m} \rangle (\frac{\nu}{\nu})^m = \frac{1}{N} \]

This shows that \( u \) and \( V \) are also complementary operators.
As before we are free to choose the phases of the components of \( \langle u_m | \) in the \( \langle u_n | \) basis

\[
\langle u_m | u_n \rangle = \frac{1}{V} \quad \text{all } n
\]

Then

\[
\langle u_m | u_n \rangle \langle u_n | u_k \rangle = \frac{1}{V} \langle u_m | u_k \rangle \quad \text{if } k \neq n
\]

\[
\frac{1}{V} \sum_{m=0}^{N-1} \langle u_m | \left( \frac{V}{V_n} \right)^{m} | u_k \rangle
\]

\[
\frac{1}{V} \sum_{m=0}^{V-1} \left( \frac{1}{V_n} \right)^{m} \langle u_m | u_k \rangle
\]

\[
\frac{1}{V} e^{-\frac{2 \pi i m k}{V}}
\]

or

\[
\langle u_n | u_k \rangle = \frac{1}{V} e^{-\frac{2 \pi i n k}{V}}
\]

\[
\langle u_k | u_n \rangle = \frac{1}{V} e^{\frac{2 \pi i n k}{V}}
\]

Note

\[
| a_n \rangle = \sum_{m=1}^{\infty} | u_m \rangle \langle u_m | a_n \rangle
\]

\[
| u_n \rangle = \sum_{m=1}^{\infty} | u_m \rangle \langle u_m | u_n \rangle
\]

\[
\langle u_m | u_n \rangle = \frac{1}{V} e^{-\frac{2 \pi i k n}{V}} = \langle u_m | a_n \rangle
\]

It follows that

\[
| a_n \rangle = | u_n \rangle
\]
This means that $A$ and $V$ have the same eigenvectors — $A$ has real eigenvalues, while $V$ has eigenvalues on the complex unit circle.

One of the nice features of $U$ and $V$ are

\[ U^m |u_n\rangle = |u_{m+n}\rangle \quad (m+n \mod N) \]

\[ \langle u_m | V^n = \langle u_{m+n} | \quad (m+n \mod N) \]

We see that in this case the complementary observables can be used to change the index on the original basis.

It follows that if we have a general operator $O$

\[ 0 = IOI = \sum_{mn=1} \langle u_m | O | u_n \rangle |u_m\rangle \langle u_n | \]

\[ = \sum_{mn=1} \langle u_m | O | u_n \rangle \cdot |u_m\rangle \langle u_n | \]
From what we have shown

\[ \langle u_m | v_n \rangle = \langle u_m | v_n \rangle^* \]

where \( m-n = m-n \mod N \). We can add or subtract as many factors of \( N \) as we need so \( m-n+kN \) is between 1 and \( N \).

This gives

\[ \langle u_m | v_n \rangle = \frac{1}{N} \sum_{k=1}^{N} (\frac{v_n}{u_m}) e^{2\pi i km} \]

This shows that \( \langle u_m | v_n \rangle \) can be expressed as a polynomial in both \( u, v \). More generally

\[ O = \sum_{m \in \mathbb{Z}} \langle u_m | v_n \rangle \frac{1}{N} e^{-\frac{2\pi i km}{N}} u^k v^{m-n} \]

Let \( l = m-n \) \( m = l+n \) \( l \equiv n \mod N \) (all numbers)

\[ \sum_{k \in \mathbb{Z}} \langle u_{l+n} | v_n \rangle \frac{1}{N} e^{-\frac{2\pi i k(l+n)}{N}} u^k v^l \]

We can define

\[ \sum_{k \in \mathbb{Z}} \langle u_{l+n} | v_n \rangle \frac{1}{N} e^{-\frac{2\pi i k(l+n)}{N}} = O_{l+n} \]

\[ O = \sum_{k \in \mathbb{R}} O_{l+n} u^k v^l \]
This is a polynomial of degree N-1 in both U and V (recall \( V^N = U^N = 1 \)) so higher powers can always be reduced to an integer between 0 and N-1.

This shows that all that we need is 2 complementary operators to represent any operator.

\[ U | V_n \rangle = | V_{n+1} \rangle \]
\[ U^\dagger | V_{n+1} \rangle = | V_n \rangle \]
\[ \langle U_n | V = \langle U_{n+1} | \]
\[ \langle U_n | = \langle U_{n+1} | V^\dagger \]

Also,
\[ \langle U^\dagger U | U_n \rangle = \langle U_{n-1} U | a \rangle^* = \langle U_{n-1} | a^\dagger \]
\[ \langle U_0 | U = \langle U_{N-1} | \]

Similarly,
\[ \langle U_{N-1} U^\dagger = \langle U_0 | \]
\[ V^\dagger U_n \rangle = | U_{n+1} \rangle \]
\[ V U_n \rangle = | U_{n-1} \rangle \]
It is also possible to write these operators as a polynomial with \( u_s \) on the right and \( u_l \) on the left:

\[
\langle u_m | u_n \rangle = \sqrt{(n-m)} \langle u_n | u_m \rangle
\]

(the exponent is mod \( n \))

\[
= \sqrt{(n-m)} \sum_{k=1}^{n} \frac{1}{n} \left( \frac{u_l}{u_n} \right)^k
\]

In this case the coefficients of the expansion will be different, this is because

\[
V U \neq U V
\]

to see the relation note

\[
\sum \langle u_n | u_l \rangle u_n \langle u_m | u_l \rangle = \sum \langle u_n | u_l \rangle u_n \langle u_m | u_l \rangle
\]

\[
\sum \langle u_n | u_l \rangle u_n \langle u_m | u_l \rangle = \sum \langle u_n | u_l \rangle u_n \langle u_m | u_l \rangle
\]

Let \( m = n + 1 \) in the sum on the right:

\[
\sum \langle u_{n+1} | u_l \rangle u_{n+1} \langle u_m | u_l \rangle = \sum e^{\frac{2\pi i}{n}} \langle u_{n+1} | u_l \rangle u_{n+1} \langle u_m | u_l \rangle
\]
comparing both expressions
\[ u^m v = v u^m e^{\frac{-2\pi i}{\nu}} \]
\[ u^n v = v u^n e^{\frac{-2\pi in}{\nu}} \]
\[ u^n v^m = v^m u^n e^{-\frac{2\pi m n}{\nu}} \]

this provides a simple means to relate the expansion
\[ 0 = \sum_{m=0}^{n-1} \sum_{m=0}^{n-1} \Omega_{mn} u^n v^m \]
\[ = \sum_{m=0}^{n-1} \Omega_{mn} e^{-\frac{2\pi m n}{\nu}} v^m u^n \]

consider a system with 3 eigenvalues
\[ B = b_1 (1 u \rangle \langle u_2 \rangle + 1 u \rangle \langle u_1 \rangle ) + b_3 1 u \rangle \langle u_3 \rangle \]

we want to write this operator as a polynomial in \( U \) and \( V \)
\[ 1 u \rangle \langle u_2 \rangle = 1 u \rangle x u_1 \rangle v = \frac{1}{3} \left( \frac{U}{e^{2\pi i/3}} \right)^3 V \]
\[ 1 u \rangle \langle u_1 \rangle = 1 u \rangle x u_2 \rangle v^2 = \frac{1}{3} \left( \frac{U}{e^{4\pi i/3}} \right)^3 V^2 \]
\[ 1 u \rangle \langle u_3 \rangle = 1 u \rangle x u_3 \rangle v = \frac{1}{3} \left( \frac{U}{e^{6\pi i/3}} \right)^3 V \]
\[ B = b_1 \left( \frac{1}{3} + \frac{1}{3} e^{\frac{-2\pi i}{3}} u + \frac{1}{3} e^{\frac{2\pi i}{3}} u^2 \right) V + \left( \frac{1}{3} + \frac{1}{3} e^{\frac{-4\pi i}{3}} u + \frac{1}{3} e^{\frac{4\pi i}{3}} u^2 \right) V^2 \]

\[ b_3 \left( \frac{1}{3} + \frac{1}{3} u + \frac{1}{3} u^2 \right) \]

We can see immediately that this is a polynomial of degree 2 in \( u \) and \( V \).

If we have 2 such operators:

1. The eigenvectors of 1 can be used as a basis for the Hilbert space.

2. The pair of operators \( U, V \) is all that is needed to describe any operator on the Hilbert space.

\((*)\) Hilbert space = vector space with complex inner product and includes limits of all convergent sequences of vectors.
Commuting observables

\* Complementary observables

If A is in state \( |a_n> \) measuring B completely destroys any information about the original state

\* Commuting observable are the opposite extreme.

If the system is put in the \( n+n \) state of A ; \( |a_n> \) and B is measured - it remains in the \( n+n \) state of A

\[ A |a_n> = a_n |a_n> \]

\[ AB |a_n> = a_n B |a_n> = BA |a_n> \]

\[ (AB-BA) |a_n> = 0 \]

\[ (AB-BA) \delta |a_n><a_n| = 0 \]

\[ AB-BA = [A,B] = 0 \]

These are called commuting observables - the relation is symmetric - measuring B does not affect A; measuring A does not affect B.
If $A$ is an observable clearly
$f(A)$ satisfies
$[A, f(A)] = 0$.

However there are some operators $B$ that are not functions of $A$ that commute.

In a system consisting of 2 silver atoms $S_{x1}$, $S_{y2}$ (the $x$ component of the spin of the first atom and the $y$ component of the spin of the second atom are commuting observables).

If $AB = BA \Rightarrow A^k B^0 = B^0 A^k$
This means for any polynomials
$P(A) Q(B) = Q(B) P(A)$

Since any function of these observables is a polynomial
$f(A) g(B) = g(B) f(A)$

For any functions $f()$ and $g()$
since
\[ |a_n\rangle \langle a_n| = P(A) \] \[ |b_m\rangle \langle b_m| = Q(B) \] both polynomials

\[ |a_n\rangle \langle a_n| \cdot |b_m\rangle \langle b_m| = |b_m\rangle \langle b_m| |a_n\rangle \langle a_n| \]

If \( |c\rangle \) is any vector

\[ |c\rangle = \sum_{mn} (a_m b_n) \langle a_m b_n| \]

In this case, we can find a basis of simultaneous eigenstates of 2 commuting observables.

Remark
\[ A |a_m b_n\rangle = a_m |a_m b_n\rangle \]
\[ B |a_m b_n\rangle = b_n |a_m b_n\rangle \]

If we define \( \Theta = F_1(A) + F_2(B) \)
\[ \Theta |a_m b_n\rangle = F_1(a_m) + F_2(b_n) \]

If \( F_1(a_m) + F_2(b_m) \) are different for
The $[\mathbf{A}, \mathbf{B}] = [\mathbf{B}, \mathbf{A}] = 0$ however here $\hat{\omega}_{m,n}$ are distinct eigenvector of $\mathbf{O}$
either products of eigenstates of $\mathbf{A}$ and $\mathbf{B}$ make a basis

eigenstates of $\mathbf{O}$ make a basis

In this case both $\mathbf{A}$ and $\mathbf{B}$ are functions of $\mathbf{O}$, but $\mathbf{O}$ is not a function of $\mathbf{A}$ or $\mathbf{B}$ alone.

A set of operators are called a complete set of commuting observables if

1. simultaneous eigenstates form a basis

2. any operator that commutes with them is a function of them.
This is an important result in quantum mechanics.

A complete measurement of a quantum system involves putting the system in a unique state labeled by the eigenvalues of a complete set of commuting observables.

The Hilbert space (vector space) spanned by the product of basis vectors of each commuting observable is a complete set of commuting observables.

I discussed how to replace products of commuting observables by a single observable with the same eigenstates.

It is also possible to break general observable into products of commuting observable.
Consider an operator A that has an eigenvalue \( \lambda \) with eigenveectors \( |a_i\rangle \) and \( |a_m\rangle \) such that

\[
|a_i\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \\
|a_{m+1}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix},
\]

We can label the states by two integers that run from 1...m,n or two with one going from 1...n and the other from 1...m.

\( |a_n\rangle \rightarrow |a_{mn}\rangle \)

Then we define two unitary operators \( U_1, U_2 \) defined by

\[
U_1 |a_{mn}\rangle = |a_{m+1,n}\rangle, \\
U_1 |a_{mn}\rangle = |a_{1,n}\rangle, \\
U_2 |a_{mn}\rangle = |a_{m,n+1}\rangle, \\
U_2 |a_{mn}\rangle = |a_{m,1}\rangle,
\]

Obviously

\[
(U_1U_2 - U_2U_1)|a_{mn}\rangle = 0
\]
\[(u_1u_2 - u_2u_1) 2 \alpha_{mn} < \alpha_{mn} = u_1u_2 - u_2u_1\]

\[\{u_1, u_2\} = 0\]

Here we see that the \(u_i\)'s that we constructed are commuting observables, since these commute we can find eigenstates \(\{| u_1, u_2\rangle\}\)

\[u_1 | u_2\rangle = u_1 | u_2\rangle\]
\[u_2 | u_1\rangle = u_2 | u_1\rangle\]

These \(u_i\) are unitary

\[| u_{i_m} u_{i_n}\rangle\]

be a basis of simultaneous eigenstates of \(u_1, u_2\)

Then we define

\[\langle u_{i_m} u_{i_n} | v_1 = \langle u_{i_m} u_{i_n} | v_2 = \langle u_{i_m} u_{i_n} | v_1\]

Just like before,

\(u_1, v_1\) are complementary

\(u_2, v_2\) are complementary

\[\{ u_1, u_2\} = \{ u_1, v_2\} = \{ v_1, v_2\} = 0\]
It is not hard to show that every operator is a polynomial in $U, U_2, V, V_2$
This decomposes a single $U \cdot V$ to pairs $U_1 \cdot V_1, U_2 \cdot V_2$
This process can be repeated until the $N$ cannot be factor

$$N = 12 = 2 \times 2 \times 3$$

$$|a_0, b_m, c_n >
U_1 \cdot U_2 \cdot U_3,

V_1 \cdot V_2 \cdot V_3$$

$U_1, V_1$ act on the first index
$U_2, V_2$ act on the second index
$U_3, V_3$ act on the third index

The result is that any quantum system with a finite number of independent states can be decomposed into products of commuting observables with prime number of eigenvalues.