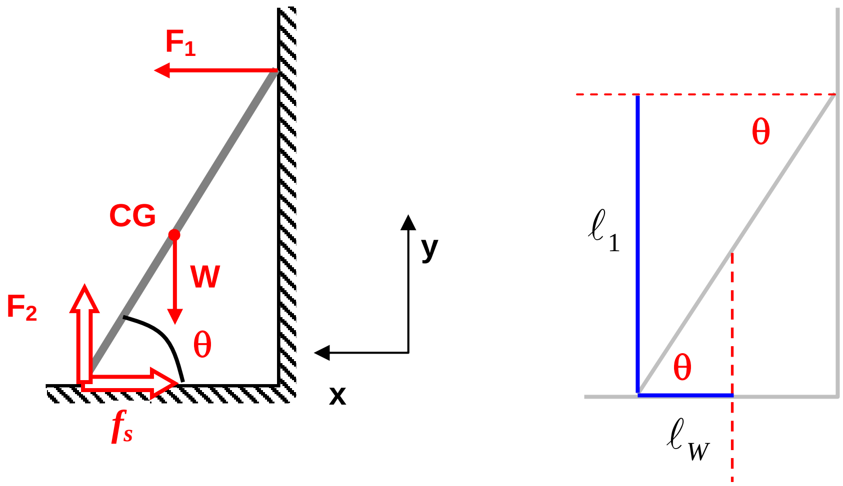
29:011 Example problem on static equilibrium of a rigid body

A uniform plank of weight W and length L rests up against a wall, making an angle θ with the horizontal. The wall is smooth so there is no friction between it and the top of the plank. The floor is rough, and the coefficient of static friction between the floor and the plank is $\mu_s = 0.40$. What is the smallest value of θ for which the plank is in static equilibrium?



For the plank to be in static equilibrium, the vector sum of the forces must be zero and the sum of all torques about a specified point must be zero. It is convenient to use the point of contact between the bottom of the plank and the floor as the point about which the torques are calculated, since then, the torques due to F_2 and f_s are zero since these forces act at that point.

Since there is no friction between the wall and the plank, the wall can only exert a normal force on the top of the plank, this is F_1. The floor is rough, so both a normal force F_2 and a static friction force f_s are exerted on the bottom of the plank by the floor.

FORCES: $\sum \vec{F} = 0 \Rightarrow \sum F_x = 0 \;\; and \;\; \sum F_y = 0$

$$\sum F_x = F_1 - f_s = 0 \rightarrow F_1 = f_s \quad (1)$$

$$\sum F_y = F_2 - W = 0 \rightarrow F_2 = W \quad (2)$$

TORQUES: $\sum \tau = 0 = F_1 \ell_1 - W \ell_W \rightarrow F_1 \ell_1 = W \ell_W \quad (3)$

Lever arms: $\ell_1 = L\sin(\theta), \quad \ell_W = (L/2)\cos(\theta) \qquad (4)$

Now $f_s = \mu_s F_2 = \mu_s W$ using (2). Using this and (4) in (3) we obtain

$$F_1 L\sin(\theta) = W(L/2)\cos(\theta) \rightarrow \text{then using (1)} \rightarrow 2f_s\sin(\theta) = W\cos(\theta)$$

$$\textit{and} \quad f_s = \mu_s W \rightarrow 2\mu_s W\sin(\theta) = W\cos(\theta)$$

$$\Rightarrow \tan(\theta) = \frac{1}{2\mu_s} = \frac{1}{2(0.40)} = 1.25 \quad \Rightarrow \boxed{\theta = 51^\circ}$$

We can look at how large the coefficient of static friction needs to be for various angles.

θ	μ_s
80	0.09
70	0.18
60	0.29
50	0.42
40	0.6
30	0.87
26.57	>1

If θ becomes too small a much larger friction force is required to keep the plank in static equilibrium.