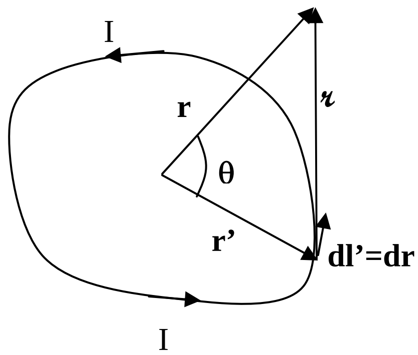
29:129 Details on the derivation of the magnetic dipole moment (12/7/12)



$$\vec{A}_{dip}\left(\vec{r}\right)=\frac{\mu_o I}{4\pi r^2}\oint_C r'\cos\left(\theta'\right)\,d\vec{\ell}$$

$$\hat{r}\cdot\vec{r}'=r'\cos\left(\theta'\right)\Rightarrow\ \vec{A}_{dip}\left(\vec{r}\right)=\frac{\mu_o I}{4\pi r^2}\oint_C\left(\hat{r}\cdot\vec{r}'\right)\,d\vec{r}'$$

consider: (i) $\hat{r}\times\left(\vec{r}'\times d\vec{r}'\right)=\vec{r}'\left(\hat{r}\cdot d\vec{r}'\right)-\left(\hat{r}\cdot\vec{r}'\right)d\vec{r}'$

$\left(\hat{r}\cdot\vec{r}'\right)d\vec{r}'=\vec{r}'\left(\hat{r}\cdot d\vec{r}'\right)-\hat{r}\times\left(\vec{r}'\times d\vec{r}'\right)$

(ii) $d\left[\left(\hat{r}\cdot\vec{r}'\right)\vec{r}'\right]=\vec{r}'\left(\hat{r}\cdot d\vec{r}'\right)+\left(\hat{r}\cdot\vec{r}'\right)d\vec{r}'$

(iii) $\oint d\left[\left(\hat{r}\cdot\vec{r}'\right)\vec{r}'\right]=0\Rightarrow\ \oint\vec{r}'\left(\hat{r}\cdot d\vec{r}'\right)=-\oint\left(\hat{r}\cdot\vec{r}'\right)d\vec{r}'$

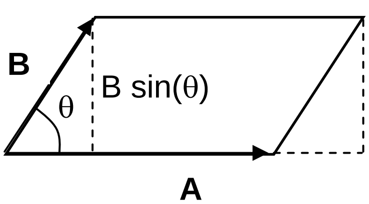
The first integral is zero since it is the integral of a perfect differential around a closed path-- the limits of integration begin and end with the same value.

then, $\oint\left(\hat{r}\cdot\vec{r}'\right)d\vec{r}'=\oint\vec{r}'\left(\hat{r}\cdot d\vec{r}'\right)-\oint\hat{r}\times\left(\vec{r}'\times d\vec{r}'\right),$

the first integral on the RHS is $-\oint\left(\hat{r}\cdot\vec{r}'\right)d\vec{r}'$,

so combining things we have that: $\oint\left(\hat{r}\cdot\vec{r}'\right)d\vec{r}'=-\frac{1}{2}\oint\hat{r}\times\left(\vec{r}'\times d\vec{r}'\right)=\left[\frac{1}{2}\oint\vec{r}'\times d\vec{r}'\right]\times\hat{r}$

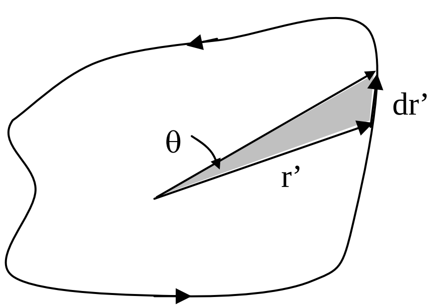
Notice that the last integral has dimensions of L^2 (area), so define $\vec{a}=\frac{1}{2}\oint\vec{r}'\times d\vec{r}'$. This is a general expression for the area of an arbitrarily shaped surface, defined as a vector. For a planar surface, *a* is just the area. To show this first consider finding the area of a parallelogram.



area $a=B\sin\left(\theta\right)A=AB\sin\left(\theta\right)=\left|\vec{A}\times\vec{B}\right|$

vector area $\vec{a}=\vec{A}\times\vec{B}$

Now apply this to an arbitrarily shaped planar surface: The area can be obtained by forming the differential triangle (shaded region): $da=(1/2)'rdr'\sin\left(\theta\right)=(1/2)\left|\vec{r}'\times d\vec{r}'\right|$. The total vector area is then $\vec{a}=\frac{1}{2}\oint\vec{r}'\times d\vec{r}'$. As an example, consider a circle of radius R:



$$a=\frac{1}{2}\oint Rd\ell=\frac{1}{2}R\oint Rd\theta=\frac{1}{2}R^2\oint d\theta$$

$$a=\frac{1}{2}R^2\cdot 2\pi=\pi R^2$$