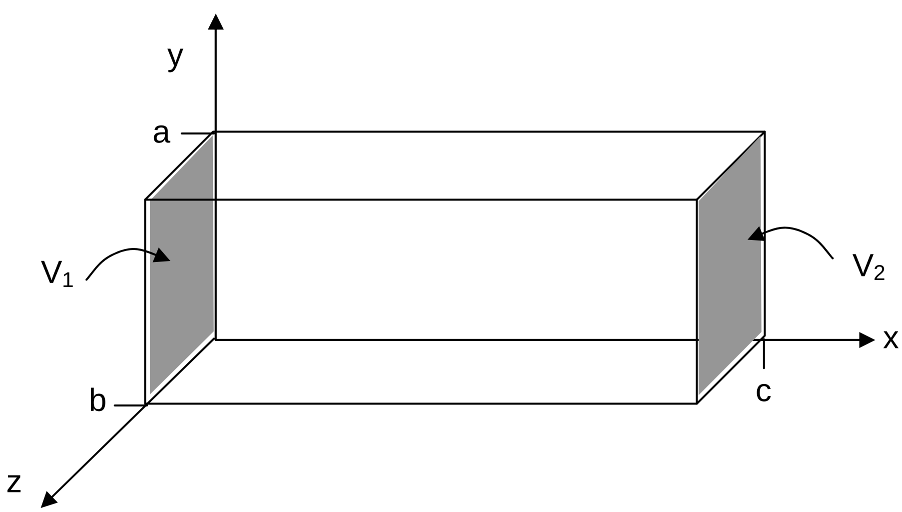
29: 129 Electricity and Magnetism I

Griffiths Example 3.5, page 134 (modified)

I will give a summary of the solution to this problem. In Griffiths' problem, the pipe is infinite in the x direction, and he gives the boundary condition on the end of the pipe (at x = 0) as $V_0(y, z)$. Since the potential on the end of the pipe is not constant, but a function of y and z, the end of the pipe cannot be a conductor, since a conductor is an equipotential. How one might go about arranging such a boundary is not clear, but e.g., one might use an insulator with different amounts of charge at different locations, or make a complicated set of little conducting squares, each insulated from each other and each having different bias voltages. We will solve a more realistic problem of a rectangular conducting box, on which the potential is specified on each of the six sides. To make the problem somewhat more general, we will close off both ends of the box and put different potentials of each end. The other 4 sides of the box will be grounded. The problem, as usual is to find the potential inside the box. We are interested not so much in the actual solution, but on the method of obtaining the solution. This problem differs from the ones that we have already considered in class in that here V = V(x, y, z), there are no ignorable coordinates. The fully 3D nature of the problem introduces some new issues that you must learn how to deal with. There is a lot of algebra that must be done to get the solution, I will not include all the steps, but give the essentials. (You will fill in the steps as part of the next homework assignment.) I try to use Griffiths' notation as much as possible.



Boundary conditions:

(1) V(x, y, b) = 0	front face
(2) V(x, y, 0) = 0	back face
(3) V(x, a, z) = 0	top
(4) V(x, 0, z) = 0	bottom
(5) $V(0, y, z) = V_1$	left face
(6) $V(c, y, z) = V_2$	right face

Following the usual separation of variable method with $V(x,y,z) = X(x)Y(y)Z(z)$, we arrive at:

$$\frac{X''(x)}{X(x)} = C_1 \quad \frac{Y''(x)}{Y(x)} = C_2 \quad \frac{Z''(x)}{Z(x)} = C_3 \quad with \quad C_1 + C_2 + C_3 = 0 \quad where \quad X''(x) = \frac{d^2X(x)}{dx^2}, \; etc.$$

The crucial step in solving boundary value problems is to determine which of the separation constants (the C's) are positive and which are negative. Here we use physical reasoning and symmetry arguments. In this particular case, we expect the solutions for Y(y) and Z(z) to be periodic, since the boundary conditions require V = 0 at y = 0 and y = a, and V = 0 at z = 0 and z = b. Clearly the x solution will not be periodic (the case with $V_1 = V_2$ is not being considered).

Recall: the second order, ordinary differential equations (the ones for X, Y, and Z) are in either of two forms:

$$F''(f) + p^2 F(f) = 0, \quad or \quad G''(g) - q^2 G(g) = 0.$$

The solutions to these ODE's are then

$$F(f) = A\sin(pf) + B\cos(pf), \quad and \quad G(g) = Ce^{qg} + De^{-qg}.$$

For the problem at hand, then we require: $C_2 = -k^2, \quad C_3 = -\ell^2, \quad and \quad C_3 = k^2 + \ell^2.$ With these then the solution for V(x, y, z) is

$$V(x,y,z) = \left(Ae^{\sqrt{k^2+\ell^2}\,x} + Be^{-\sqrt{k^2+\ell^2}\,x}\right)\left[C\sin(ky) + D\cos(ky)\right]\left[F\sin(\ell z) + G\cos(\ell z)\right].$$

Now we note here that since x is limited to 0 – c, both A and B will be needed. (In the example 3.5 that Griffiths is solving, the right end is open so x goes to infinity and to insure that V goes to zero when x goes to infinity, he must require that A = 0.) Applying boundary conditions (1) – (4) we require:

$$BC\ (4) \rightarrow D = 0 \qquad BC\ (3) \rightarrow ka = n\pi \rightarrow \quad k = n\pi/a$$
$$BC\ (2) \rightarrow G = 0 \qquad BC\ (1) \rightarrow \ell b = m\pi \rightarrow \quad \ell = m\pi/b$$
$$and \quad C_3 = \pi\sqrt{(n/a)^2 + (m/b)^2}$$

The solution then simplifies to

$$V(x,y,z) = \left[Ae^{\pi\sqrt{\left(\frac{n}{a}\right)^2+\left(\frac{m}{b}\right)^2}\,x} + Be^{-\pi\sqrt{\left(\frac{n}{a}\right)^2+\left(\frac{m}{b}\right)^2}\,x}\right] C\sin\left(\frac{n\pi y}{a}\right) F\sin\left(\frac{m\pi z}{b}\right).$$ Now, boundary conditions

(5) and (6) cannot be satisfied by a simple choice for A, B, C, and F, but we realize that with n and m

as integers from 0 to infinity, that a general solution can be obtained by adding all the solutions, which in this case amounts to a double summation over n and m,

$$V(x,y,z)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{\left[P_{n,m}e^{\pi\sqrt{\left(\frac{n}{a}\right)^2+\left(\frac{m}{b}\right)^2}\,x}+Q_{n,m}e^{-\pi\sqrt{\left(\frac{n}{a}\right)^2+\left(\frac{m}{b}\right)^2}\,x}\right]\sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{m\pi z}{b}\right)\right\}, \qquad (1)$$

where all the constants have been absorbed into P and Q. This mess is a double Fourier series. The trick now is to get P and Q, using the orthogonality of the sine functions, which form a complete set.

The procedure is to put V(0,y,z) = V_1 in (1) multiply both sides of the last equation by $\sin\left(\frac{n'\pi y}{a}\right)\sin\left(\frac{m'\pi z}{b}\right)$ and integrate over y from 0 to a, and over z from 0 to b, then put V(c,y,z) = V_2 in (1) and multiply both sides of the last equation by $\sin\left(\frac{n'\pi y}{a}\right)\sin\left(\frac{m'\pi z}{b}\right)$ and integrate over y from 0 to a, and over z from 0 to b The right hand side will collapse due to the orthogonality of the sine functions. The required integrals are

$$\int_0^a \sin\left(\frac{n'\pi y}{a}\right)\sin\left(\frac{n\pi y}{a}\right)dy=\frac{a}{2}\delta_{n'n} \qquad \int_0^b \sin\left(\frac{m'\pi y}{b}\right)\sin\left(\frac{m\pi y}{a}\right)dy=\frac{b}{2}\delta_{m'm}$$

(remember that $\delta_{ik}=1,\ \ when\ \ i=j,\ \ and\ \ 0\ \ when\ \ i\neq j$). The effect of these integrals is to collapse the sum over *n* to one term in n' and the sum over *m* to one term in m'. The integrals from the left hand side of the resulting equation are simple integrals over a sine function and the result is

$$\{V_1\ or\ V_2\}\times\int_0^b\int_0^a \sin\left(\frac{n'\pi y}{a}\right)\sin\left(\frac{m'\pi y}{b}\right)dydz=\{V_1\ or\ V_2\}\times\frac{2a}{n'\pi}\frac{2b}{m'\pi},\ \ n',m'=1,3,5,\ldots \qquad (2)$$

This will give 2 equations to determine P and Q:

$$\frac{4abV_1}{\pi^2 n'm'}=\frac{ab}{4}\left[P_{n',m'}+Q_{n',m'}\right]\ \ (3)\quad \text{and} \quad \frac{4abV_2}{\pi^2 n'm'}=\frac{ab}{4}\left[P_{n',m'}e^{\pi\sqrt{\left(\frac{n}{a}\right)^2+\left(\frac{m}{b}\right)^2}\,c}+Q_{n',m'}e^{-\pi\sqrt{\left(\frac{n}{a}\right)^2+\left(\frac{m}{b}\right)^2}\,c}\right] \qquad (4).$$

(3) and (4) can be solved to obtain $P_{n,m}$ and $Q_{n,m}$ (we can drop the primes now for convenience),

$$P_{n,m} = \frac{16}{\pi^2 nm}\left[\frac{\left(V_2 - V_1 e^{-\pi\sqrt{\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2}\,c}\right)}{e^{\pi\sqrt{\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2}\,c} - e^{-\pi\sqrt{\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2}\,c}}\right], \quad \text{and} \tag{5a}$$

$$Q_{n,m} = \frac{16}{\pi^2 nm}\left[\frac{\left(V_1 e^{\pi\sqrt{\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2}\,c} - V_2\right)}{e^{\pi\sqrt{\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2}\,c} - e^{-\pi\sqrt{\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2}\,c}}\right]. \tag{5b}$$

The expressions for P and Q look rather complicated, but remember they depend only on the given constants of the problem, a, b, c, V_1 and V_2 and the summation indices n and m.

Put (5a) and (5b) into (1) and you have the solution. Remember, the sums are over *n*, *m* = 1, 3, 5, . . .