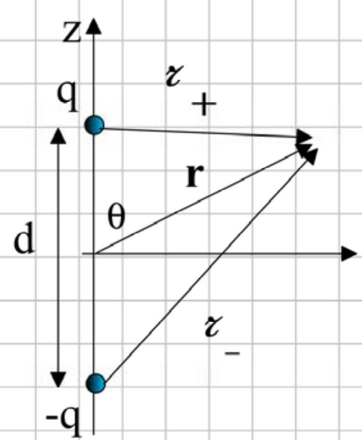
# Multipole expansion

> *restart*;

For the **electric dipole,** with $x = \frac{d}{r}$, where d is the separation between the charges.
Compute V(r) for r >> d.




$$r_{\pm} = \sqrt{r^2 + \left(\frac{d}{2}\right)^2 \mp 2\,r\,\frac{d}{2}\cos(\theta)}$$

$$\frac{1}{r_{\pm}} = \frac{1}{r}\left[1 + \frac{d^2}{r^2} \mp \frac{d}{r}\cos(\theta)\right]^{-\frac{1}{2}}$$

> $g := \left(1 + x^2 - x\cdot\cos(\theta)\right)^{-\frac{1}{2}} - \left(1 + x^2 + x\cdot\cos(\theta)\right)^{-\frac{1}{2}}$;

$$g := \frac{1}{\sqrt{1 + x^2 - x\cos(\theta)}} - \frac{1}{\sqrt{1 + x^2 + x\cos(\theta)}} \quad \textbf{(1)}$$

> *taylor*$(g, x, 6)$;

$$\cos(\theta)\,x + \left(-\frac{3}{2}\cos(\theta) + \frac{5}{8}\cos(\theta)^3\right)x^3 + \left(\frac{15}{8}\cos(\theta) - \frac{35}{16}\cos(\theta)^3 + \frac{63}{128}\cos(\theta)^5\right)x^5 + O(x^7) \quad \textbf{(2)}$$

The leading order term (the largest term in the series is: $\frac{d}{r}\cos(\theta)$, so that the potential is

$$V(r, \theta) = \frac{q}{4\,\pi\varepsilon_o}\left(\frac{1}{r_+} - \frac{1}{r_-}\right) \approx \frac{q}{4\,\pi\varepsilon_o}\;\frac{1}{r}\;\frac{\cos(\theta)\,d}{r}$$

$$= V(r, \theta) \approx \frac{qd}{4\,\pi\varepsilon_o r^2}\cos(\theta) = \frac{p}{4\,\pi\varepsilon_o r^2}\cos(\theta).$$

We define the electric dipole moment **p** as a vector of length qd pointing from the
the
negative to the positive charge, then $\boldsymbol{p} = qd\,\boldsymbol{u_z}$ *so that* $\boldsymbol{p.u_r} = p\cos(\theta)$,

and $V_{dip}(r, \theta) = \dfrac{\boldsymbol{p.u_r}}{4\pi\varepsilon_o r^2}$ . $u_z$ **and** $u_r$ are the z and r unit vectors

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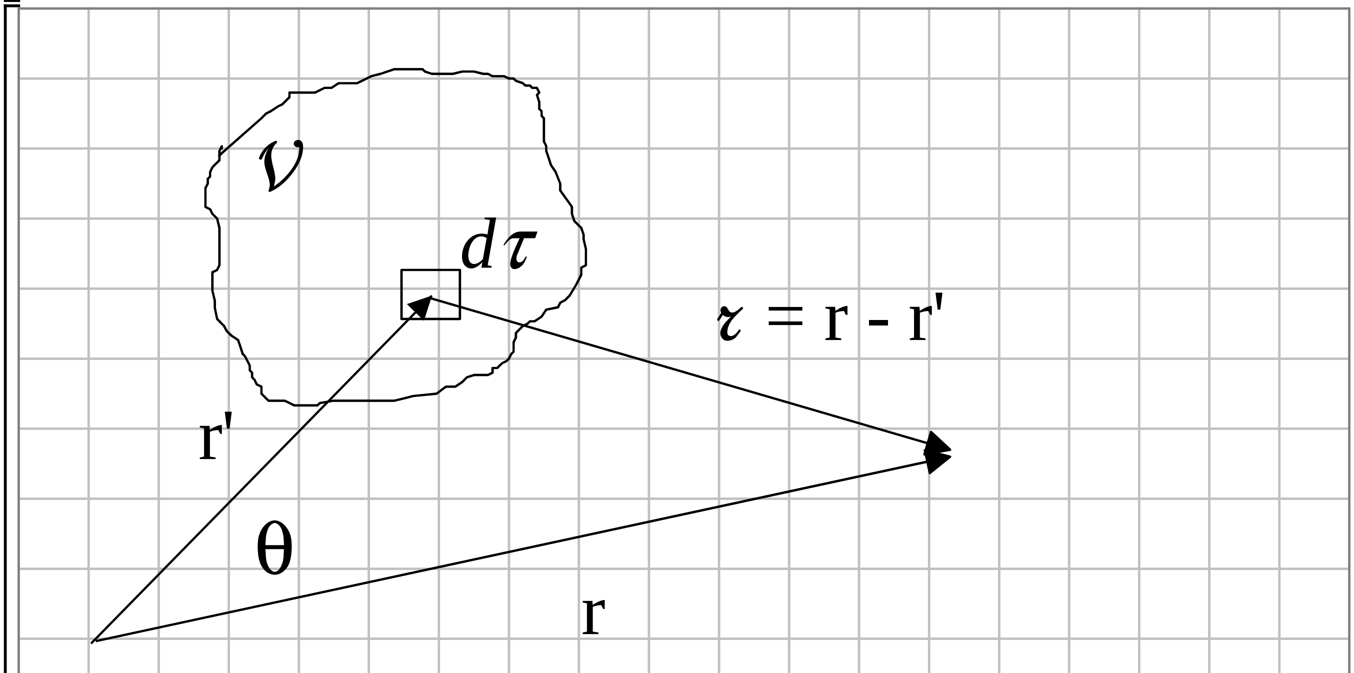
For a general 3D bounded charge distribution:

$\rho(r')$, $V(\boldsymbol{r}) = \dfrac{1}{4\pi\varepsilon_o}\displaystyle\int_{\mathcal{V}} \dfrac{\rho(\boldsymbol{r}')}{\mathscr{r}}\, d\tau$




$$\mathscr{r}^2 = r^2 + r'^2 - 2rr'\cos(\theta)$$

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\left[1 + \frac{r'^2}{r^2} - \frac{2r'}{r}\cos(\theta)\right]^{-\frac{1}{2}}$$

with $x = \dfrac{r'}{r} < 1$

> $f := \left(1 + x^2 - 2 \cdot x \cdot \cos(\theta)\right)^{-\frac{1}{2}};$

$$f := \frac{1}{\sqrt{1 + x^2 - 2\, x \cos(\theta)}} \qquad \textbf{(3)}$$

> $taylor(f, x, 8);$

$$1 + \cos(\theta)\, x + \left(-\frac{1}{2} + \frac{3}{2}\cos(\theta)^2\right) x^2 + \left(-\frac{3}{2}\cos(\theta) + \frac{5}{2}\cos(\theta)^3\right) x^3 + \left(\frac{3}{8} - \frac{15}{4}\cos(\theta)^2 + \frac{35}{8}\cos(\theta)^4\right) x^4 + \left(\frac{15}{8}\cos(\theta) - \frac{35}{4}\cos(\theta)^3 + \frac{63}{8}\cos(\theta)^5\right) x^5 + \left(-\frac{5}{16} + \frac{105}{16}\cos(\theta)^2 - \frac{315}{16}\cos(\theta)^4 + \frac{231}{16}\cos(\theta)^6\right) x^6 + \left(-\frac{35}{16}\cos(\theta) + \frac{315}{16}\cos(\theta)^3 - \frac{693}{16}\cos(\theta)^5 + \frac{429}{16}\cos(\theta)^7\right) x^7 + \mathrm{O}(x^8) \qquad \textbf{(4)}$$

> $with(orthopoly):$

> **for** $n$ **from** 1 **by** 1 **to** 7 **do** $P(n, \cos(\theta))$ **od**;

$$\cos(\theta)$$

$$-\frac{1}{2} + \frac{3}{2}\cos(\theta)^2$$

$$-\frac{3}{2}\cos(\theta) + \frac{5}{2}\cos(\theta)^3$$

$$\frac{3}{8} - \frac{15}{4}\cos(\theta)^2 + \frac{35}{8}\cos(\theta)^4$$

$$\frac{15}{8}\cos(\theta) - \frac{35}{4}\cos(\theta)^3 + \frac{63}{8}\cos(\theta)^5$$

$$-\frac{5}{16} + \frac{105}{16}\cos(\theta)^2 - \frac{315}{16}\cos(\theta)^4 + \frac{231}{16}\cos(\theta)^6$$

$$-\frac{35}{16}\cos(\theta) + \frac{315}{16}\cos(\theta)^3 - \frac{693}{16}\cos(\theta)^5 + \frac{429}{16}\cos(\theta)^7 \qquad \textbf{(5)}$$