

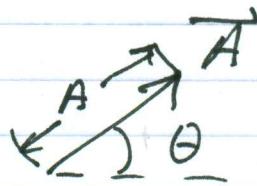
29:129

ORTHOGONAL FUNCTIONS & EXPANSIONS

- extension of the concept of vectors in 3D to vectors of infinite dimensions

3D VECTORS

$$\vec{A} = (A, \theta)$$



Introduce "basis vectors" $\hat{x}, \hat{y}, \hat{z}$

with properties: $\hat{x} \cdot \hat{x} = \hat{y} \cdot \hat{y} = \hat{z} \cdot \hat{z} = 1$

$$\hat{x} \cdot \hat{y} = \hat{y} \cdot \hat{z} = \hat{z} \cdot \hat{x} = 0$$

1st property \Rightarrow basis is normalized

2nd property \Rightarrow basis vectors are orthogonal (1)

Note that the operation - scalar (or dot) product must be defined as part of the specification of basis.

2nd property also implies linear independence

$\hat{x} \neq c\hat{y}$ etc - cannot express

one basis vector as a linear combination of other basis vectors.

3rd property - basis is Complete

$$\vec{A} = A_1 \hat{x} + A_2 \hat{y} + A_3 \hat{z}$$

\Rightarrow linear combination

To get: A_1, A_2, A_3 :

$$\vec{A} \cdot \hat{x} = A_1, \vec{A} \cdot \hat{y} = A_2, \vec{A} \cdot \hat{z} = A_3$$

Orthonormal Functions

$A(x)$ is a "vector" of infinite dimensions

Define scalar product of $A(x) \neq B(x)$

$$(A, B) = \int_a^b A(x) B(x) dx \text{ on } (a, b)$$

Orthogonality - 2 vectors are orthog.

$$\text{if } (A, B) = 0$$

Normalized vectors

$A(x)$ is Normalized if

$$(A, A) = \int_a^b A(x) A(x) dx = \int_a^b [A(x)]^2 dx = 1$$

Consider a set of functions defined as

$$f_k(x) \quad k=1, 2, 3, \dots$$

$f_k(x)$ are ORTHONORMAL if

$$\int_a^b f_k(x) f_l(x) dx = \begin{cases} 0 & k \neq l \\ 1 & k = l \end{cases}$$

Then f_k 's are linearly independent

$$(f_k, f_l) = \delta_{kl} \quad (=0 \text{ if } k \neq l, =1 \text{ if } k=l)$$

A linearly independent, orthonormal set can be used as a basis in which to expand other functions

$$\text{e.g. } g(x) = \sum_{k=1}^{\infty} c_k f_k(x)$$

Must determine c_k 's, must $\times f_k(x)$ and integrate

$$\int_a^b f_k(x) g(x) dx = \sum_k c_k \int_a^b f_k(x) f_k(x) dx$$

$$\int_a^b f_\ell(x) g(x) dx = \sum c_k \delta_{\ell k}$$

$$\int_a^b f_\ell(x) g(x) dx = C_\ell$$

Solutions to $\nabla^2 V = 0$ -

∇^2 is a linear operator -

if we have solutions V_1, V_2, \dots etc.

Then any linear combination of

these solutions is also a solution.

Proof $V = a_1 V_1 + a_2 V_2 + \dots + a_n V_n + \dots$
 a_i 's = const.

$$\nabla^2 V = a_1 \nabla^2 V_1 + a_2 \nabla^2 V_2 + \dots + a_n \nabla^2 V_n + \dots$$

$$\text{But } \nabla^2 V_n = 0, \text{ so}$$

$\nabla^2 V = 0$, $V = \sum c_n V_n$ is a
SOLUTION