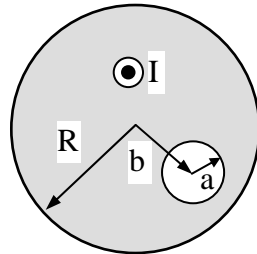
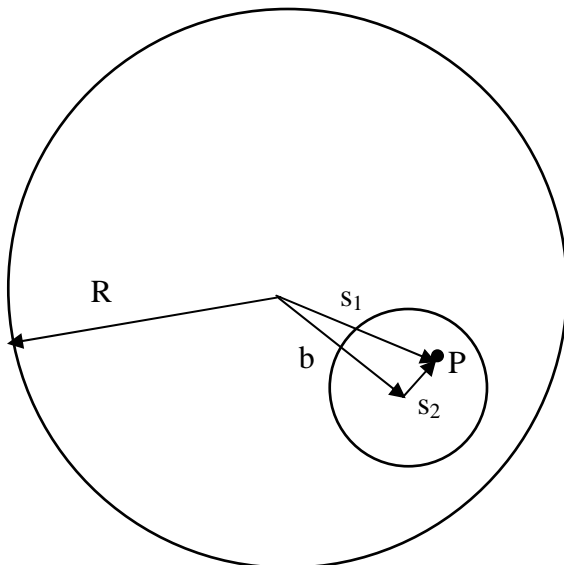
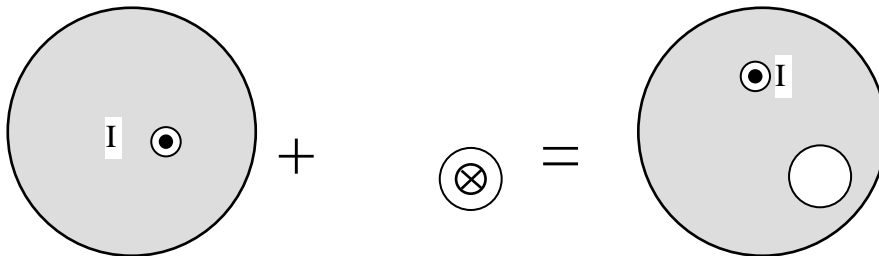


## A classic magnetostatics problem involving Ampere's Law and the superposition principle

A long straight conductor has a circular cross section of radius  $R$  and carries a current  $I$ . Inside the conductor, there is a cylindrical hole of radius  $a$  whose axis is parallel to the axis of the conductor and at a distance  $b$  from it. The problem is to find the magnetic field inside the hole and show that it is uniform.



Now since the hole destroys all symmetry in the configuration, Ampere's law cannot be applied directly to solve this problem. Also trying to apply the Biot-Savart Law would be hopeless. The trick to solving this problem is to use the principle of superposition in a clever way. If the hole were no present, Ampere's law could be used to compute  $B$  inside the conductor. We imagine that there is no hole and the current is distributed uniformly throughout the entire cross section of the conductor. Then, imagine that another current of the same current density but of opposite direction is superposed in the space occupied by the hole, so that there is no net current in the hole. Then we get  $B$  inside the hole by superposition of the fields of the two currents. Schematically:



We will compute the field at point  $P$  inside the hole. Point  $P$  is a distance  $s_1$  from the center of the large cylinder and a distance  $s_2$  from the center of the hole.

In superposing the currents the current densities must be equal but in opposite directions. The current density  $J$  is given by

$$J = \frac{I}{\pi R^2 - \pi a^2}.$$

The magnitudes of the fields  $B_1$  and  $B_2$  at P due to the two currents,  $J_1$  in the conductor, and  $J_2$  in the hole (where  $\vec{J}_1 = \vec{J}$  and  $\vec{J}_2 = -\vec{J}$ ) are obtained by a direct application of Ampere's Law:

$$\oint_C \vec{B} \cdot d\vec{\ell} = \mu_o I_C \rightarrow B_1(2\pi s_1) = \mu_o J_1 \pi s_1^2 \quad \text{and} \quad B_2(2\pi s_2) = \mu_o J_2 \pi s_2^2$$

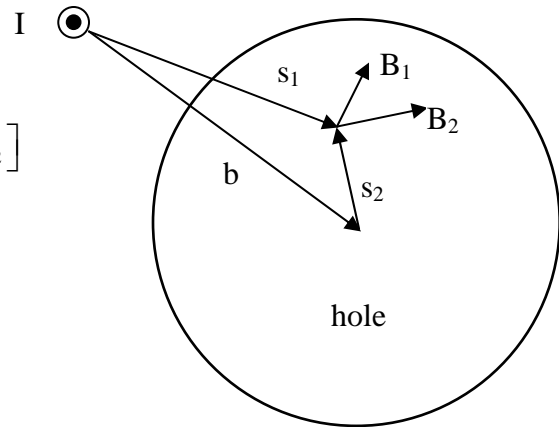
$$\rightarrow B_1 = \frac{\mu_o J_1 s_1}{2}, \quad B_2 = \frac{\mu_o J_2 s_2}{2}$$

Now  $B_1$  and  $B_2$  are not in the same direction, so the magnetic field vectors must be superposed in the hole region  $\vec{B} = \vec{B}_1 + \vec{B}_2$ . The diagram shows an enlarged view of the hole with the various vectors involved. Since both  $J_1$  and  $J_2$  are cylindrical conductors, the magnetic field lines are circles centered on the current axes. The direction of the magnetic fields are perpendicular to the vectors  $\vec{s}_1$  and  $\vec{s}_2$  in the direction specified by the right hand rule. A convenient way to specify the magnetic field direction for cylindrical conductors (or straight wires) is that  $\vec{B}$  points in the direction given by  $\vec{J} \times \vec{s}$ . Since  $J$  and  $s$  are perpendicular, the magnitude is just  $J s$ . Then

$$\vec{B}_1 = \frac{\mu_o}{2} \vec{J}_1 \times \vec{s}_1 \quad \text{and} \quad \vec{B}_2 = \frac{\mu_o}{2} \vec{J}_2 \times \vec{s}_2, \quad \text{then}$$

$$\vec{B} = \vec{B}_1 + \vec{B}_2 = \frac{\mu_o}{2} [\vec{J}_1 \times \vec{s}_1 + \vec{J}_2 \times \vec{s}_2] = \frac{\mu_o}{2} [\vec{J} \times \vec{s}_1 - \vec{J} \times \vec{s}_2]$$

$$\rightarrow \vec{B} = \frac{\mu_o}{2} \vec{J} \times (\vec{s}_1 - \vec{s}_2)$$



We see from the diagram that

$$\vec{b} + \vec{s}_2 = \vec{s}_1 \rightarrow \vec{s}_1 - \vec{s}_2 = \vec{b},$$

so that  $\vec{B} = \frac{\mu_o}{2} \vec{J} \times \vec{b}$ . This gives  $\vec{B}$  anywhere in the hole. Finally, expressing  $J$  in terms of  $I$ , we find the magnitude of  $B$  in the hole as

$$B = \frac{\mu_o I b}{2\pi(R^2 - a^2)}, \quad \text{or in vector form}$$

$$\vec{B} = \frac{\mu_o \vec{I} \times \vec{b}}{2\pi(R^2 - a^2)}$$

$\vec{B}$  is uniform in the hole!