29:28 A very brief introduction to complex variables

Real numbers:

$x^2 - 4 = 0 \rightarrow x = \pm 2$

Imaginary numbers:

$x^2 + 1 = 0$, $x = \pm\sqrt{-1} = \pm i$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, etc…

Complex numbers:

$$z^2 - 2z + 2 = 0 \rightarrow z = \frac{-2 \pm \sqrt{4-(4)(2)}}{2} = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i$$

A complex number, z, is a pair of <u>real numbers</u>, x and y, of the form

$$z = x + i\,y \qquad (1)$$

The real part of z is x and the imaginary part is y.

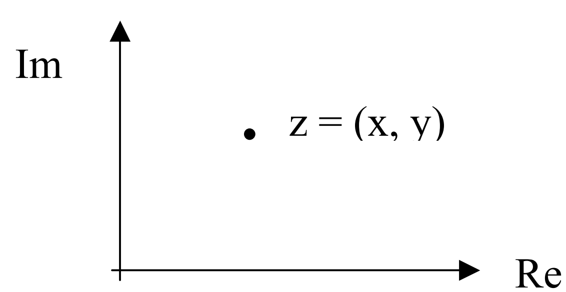
Euler's relation:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta), \qquad (2a)$$

$$e^{-i\theta} = \cos(\theta) - i\sin(\theta), \qquad (2b)$$

Argand diagram

A complex number can be plotted as a point in a *complex plane*. The real part of the complex number is plotted on the horizontal axis and the imaginary part of the vertical axis. The point (x,y) represents the complex number.

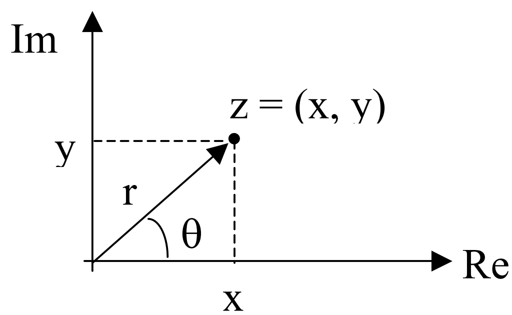


Polar form of a complex number

Another convenient way to represent cpmplex munbers is in polar form, namely

$$z = r(\cos\theta + i \sin\theta) \qquad (3)$$

where $x = r \cos\theta$ and $y = r \sin\theta$. Also $r = |z| = \sqrt{x^2 + y^2}$ and $\tan\theta = y/x$.



Using Euler's relation, (2a), the polar form can also be written as

$$z = r\,e^{i\theta}. \qquad (4)$$

Complex numbers in physics

Complex numbers are especially useful in physics to represent oscillating quantities. For example a the position of a simple harmonic oscillator is usually expressed as

$$x(t) = A \cos(\omega t + \varphi) \qquad (5)$$

where A is the amplitude of the motion, $\omega = 2\pi f$ is the angular frequency, and φ is the phase angle. The phase angle φ is introduced to allow for various initial conditions. From equation (3) we see that $r\cos\theta$ is the real part of the complex number z. Thus we can represent x(t) as the real part of a complex number also:

$$x(t) = \mathrm{Re}\{Ae^{i(\omega t + \varphi)}\} = \mathrm{Re}\{ Ae^{i\varphi}\, e^{i\omega t}\} \qquad (6)$$

where Re { } means take the real part of the stuff in { }. Physical quantities must always be real. We can use complex quantities to represent physical quantities if we only use the real part of the complex quantity. Often, we do not even use the Re{ } notation, but just keep in mind that it is the real part that is physical, so we will write:

$$x(t) = Ae^{i(\omega t + \varphi)} \tag{7}$$

This form, (7) is particularly useful because it separates the phase angle information from the time dependent part. This is very convenient when computing derivatives (or integrals) of quantities since the operation does not change cosines into sines [the derivative and integral of e^x is e^x!]. For example, if we want the velocity of the oscillator then from (7)

$$\begin{aligned} v = \frac{dx}{dt} = i\omega Ae^{i(\omega t+\varphi)} &= i\omega A[\cos(\omega t + \varphi) + i\sin(\omega t + \varphi)] \\ &= \omega A[i\cos(\omega t + \varphi) - \sin(\omega t + \varphi)] \end{aligned} \tag{8}$$

The real part of this (the physical velocity) is

$$v = -\omega A\sin(\omega t + \varphi)\,. \tag{9}$$

Notice that this is the same result that you would obtain by taking the time derivative of (5).

Complex numbers are also useful in representing currents and voltages in AC circuits, since these quantities have amplitudes and phase. In this case, the complex notation can be used to represent phasors.