Anomalous Diffusion Arising from Microinstabilities in a Plasma

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A plasma is considered in which a Maxwellian distribution of electrons with thermal velocity \( v_e \) and drift velocity \( v_d \) is drifting relative to a Maxwellian distribution of ions with thermal velocity \( v_i \).

The rate at which the ions drift, \( v_d \), is parallel to the electric field, \( E \), and the magnetic field, \( B \), is perpendicular to both. The electron density, \( n_e \), is constant and the ion density, \( n_i \), is the sum of a Maxwellian distribution of ions with thermal velocity \( v_i \), and a streaming component with velocity \( v_d \) parallel to the electric field.

In the case when \( \beta = 2 \frac{v_d}{v_i} \), the electron Larmor radius and \( \Omega_e \) is the electron Larmor frequency. \( \alpha \), the ratio of the resulting diffusion coefficient to the Bohm diffusion coefficient, is given by a constant \( \alpha \).

\[ \alpha = \frac{v_d}{v_i} \left( \frac{T_e}{T_i} \right)^2 \]

I. INTRODUCTION

It is well known that a fluctuating electric field in a plasma leads to a diffusion of the particles across the magnetic field, and this is called anomalous diffusion to distinguish it from the usual collisional diffusion. This diffusion can be divided into two classes which depend on the character of the fluctuating electric fields. We denote the \( k \)th Fourier space component of the electric field by

\[ E_k = \hat{\phi}_k \exp[i\phi_k(t)], \]

where \( \hat{\phi}_k \) is a unit vector in the direction of polarization. In one case the electric field is characterized by a phase \( \phi_k(t) \), which is a statistical function of time and as discussed by Spitzer as the diffusion coefficient depends on the correlation time of the electric field fluctuations.

In the second case the electric field is made up of a superposition of coherent waves and

\[ \phi_k(t) = -\omega_k t + \phi_{ks}, \]

where \( \phi_{ks} \) is independent of time. In this case the diffusion arises from a resonance of particles and waves which move along the field lines with the same velocity.

It has been shown that in the nonlinear limit certain types of microinstabilities lead to the establishment of an equilibrium electric fluctuation spectrum which is a superposition of coherent waves. It is thus possible to determine the anomalous diffusion arising from these microinstabilities by direct calculation and it is the purpose of this paper to do this for the case of a particularly strong instability—the two-stream ion-cyclotron instability.

In Sec. II we develop the linearized theory of the two-stream ion-cyclotron instability; the nonlinear theory of this instability is discussed in Sec. III. The diffusion due to the nonlinear equilibrium spectrum is calculated in Sec. IV and the results are discussed in Sec. V.

II. LINEARIZED THEORY

We consider a homogeneous infinite collisionless plasma in which the ions and electrons each have a Maxwellian distribution at a characteristic temperature with the center of the Maxwellians displaced by a drift velocity \( v_d \) and we restrict ourselves to the case of \( T_e \approx T_i \) and \( \beta = \frac{8\pi neKT_i}{B^2} \ll 1 \). (Here \( T_e \) and \( T_i \) are the electron and ion temperatures respectively, \( n \) is the particle density, \( B \) is the magnetic field strength, and the drift velocity is along the magnetic field.) The usual theory of the two-stream instability, which considers only waves propagating parallel to the magnetic field, would predict stability until the electron drift velocity becomes comparable to the electron thermal velocity. Only in situations where \( T_e \gg T_i \) does the critical velocity approach the ion thermal velocity. In the following we point out that for a collisionless plasma instability occurs at a much lower velocity for electrostatic waves near the ion cyclotron frequency and propagating at large angles to the field.

We will work in the frame where the ions are at rest and consider modes of the form \( \exp(i\mathbf{k} \cdot \mathbf{r} - \omega t) \).

If \( v_d = 0 \) the plasma is evidently stable. We would therefore expect that instability could only occur if \( kT_e > \omega \) where \( \omega \) is the wave frequency and \( k_{\parallel} \) the wavenumber parallel to the field. This is the condition that the peak of electron distribution be moving slightly faster than the wave—the usual
condition for being able to put energy into the wave.

In this case if we put \( \omega \approx \Omega_i \), the ion-cyclotron frequency, and \( v_i < v_p \ll v_e \), the case we wish to discuss, we obtain the condition \( k_p \Omega_i \approx 1 \), where \( v_i \) and \( v_e \) are the electron and ion thermal velocities respectively and \( \rho_i = v_i/\Omega_i \) is the ion Larmor radius. The significance of this is that we can show that for such \( k \) and \( \omega \) both pure electrostatic waves are possible. Thus if we write down the dispersion matrix obtained from \( \nabla \times \nabla \times \mathbf{E} + \mathbf{\tilde{E}} = -4\pi \mathbf{j} \) in a coordinate system in which one of the axes is parallel to \( k \), determining \( \mathbf{j} \) from the Boltzmann equation, we find the dispersion matrix has the following structure:

\[
\begin{bmatrix}
  c k^2 - \omega^2 + (\alpha_{11}) & (\alpha_{12}) & (\alpha_{13}) \\
  (\alpha_{21}) & c k^2 - \omega^2 + (\alpha_{22}) & (\alpha_{23}) \\
  (\alpha_{31}) & (\alpha_{32}) & -\omega^2 + (\alpha_{33})
\end{bmatrix} = 0,
\]

(1)

where the quantities \( \alpha_{ij} \) arise from the plasma currents and are all of order \( \omega^2/(k\lambda_p)^2 \), where \( \lambda_{pi} \approx k_{pi} \Omega_i/\omega \). We note that for \( c k^2 \gg \omega^2 \), \( \omega^2/(k\lambda_p)^2 \), the only possible root is \( \omega^2 = (\alpha_{33}) \), the pure electrostatic mode in which \( \mathbf{E} \parallel \mathbf{k} \). If we put \( \omega \approx \Omega_i \), \( k \approx \Omega_i/v_i \), then \( (k\lambda_p)^2 \approx B^2/4\pi n m_e e^2 \ll 1 \) and \( c k^2 (k\lambda_p)^2/\omega^2 \approx B^2/4\pi n m_e v_i^2 = 1/\beta \gg 1 \). Thus we need only consider pure electrostatic modes.

For this case the dispersion relation, which is worked out in the Appendix, is

\[
\sum_{j = 1}^{n} \sum_{\alpha} \Gamma_j \frac{k_{\alpha}^2}{(k\lambda_p)^2} \left\{ \frac{\omega + k_j u_j + n \Omega_i}{k \omega_i} - \frac{n \Omega_i}{\omega + k_j u_j + n \Omega_i} \right\} = 1.
\]

(2)

Here \( T_j \) is temperature, \( u_j \) the drift velocity of the \( j \) species, \( \Gamma_j(x) = e^{-x} I_n(x) \), \( I_n \) is the usual Bessel function of imaginary argument, and

\[
W(x) = -1 + \frac{x}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \frac{\exp \left( \frac{1}{2} y^2 \right)}{x + y} \, dy.
\]

(3)

The contour of the integral may be taken along the real axis for \( x \) in the lower half-plane (growing waves). Limiting forms are given by

\[
W \approx -1 + i \left( \frac{1}{2} \right)^{1/2} x, \quad |x| < 1,
\]

(4)

\[
W \approx 1/x^2, \quad |x| \gg 1.
\]

For all real \( x \), \( \text{Im} \, W = i \left( \frac{1}{2} \right)^{1/2} x \exp \left( -\frac{1}{2} x^2 \right) \). The limiting form given for large \( x \) is not valid for highly damped waves which are of no concern here.

Since we are concerned with wavelengths comparable to the ion-gyroradius \( k_{i\rho} \ll 1 \); \( \Gamma_{i\omega} \approx 1 \); \( \Gamma_{i\omega} \approx 0 \). Moreover we see that since \( v_p/v_e \ll 1 \) the argument of the electron \( W \) function is very small. Since we are concerned with a wave nearly at resonance with the ion-gyrofrequency we may neglect all the ion terms except \( n = 1 \). This gives us

\[
0 = \left[ -1 + \left( \frac{1}{2} \right)^{1/2} \left( \frac{-\omega + k_j \omega_p}{k_j \omega_i} \right) \right] + \left( \frac{T_j}{T_i} \right) \Gamma_1 \left[ \frac{W\left( \frac{-\omega + \Omega_i}{k_j \omega_i} \right)}{W\left( \frac{-\omega + \Omega_i}{k_j \omega_i} \right)} \right],
\]

(5)

where we have neglected \( (k\lambda_p)^2 \ll 1 \).

We may obtain an approximate solution by noting that a condition for solution is a large argument for the ion \( W \) function as otherwise the large imaginary part (ion cyclotron damping) will give a damped solution.

If

\[
|\omega - \Omega_i|/k_j \omega_i > 1,
\]

(6)

we have simply

\[
\frac{\Omega_i}{\omega - \Omega_i} T_j T_i \Gamma_1 \left[ 1 - \left( \frac{1}{2} \right)^{1/2} \left( \frac{-\omega + k_j \omega_p}{k_j \omega_i} \right) \right],
\]

(7)

\[
\omega - \Omega_i = \Omega_i T_j T_i \Gamma_1 \left[ 1 + \left( \frac{1}{2} \right)^{1/2} \left( \frac{-\omega + k_j \omega_p}{k_j \omega_i} \right) \right].
\]

(8)

Here we have chosen \( k_i \) positive as the direction of propagation for instability and used \( \omega \approx \Omega_i \).

We note that \( \Gamma_1 \) has a very flat maximum at \( k_i^2 R_L^2 \approx 1.5 \) attaining there a value 0.22.

We conclude therefore that the maximum growth rate is given by

\[
\gamma \approx 0.3(T_j/T_i) \Omega_i (v_p/v_e)
\]

occuring for \( k_i^2 R_L^2 \approx 1.5 \) and \( k_i \approx \Omega_i/v_p \).

Moreover from Eq. (6) we must have

\[
\frac{\omega - \Omega_i}{\Omega_i/\omega_j} \approx 0.2 \left( \frac{T_j}{T_i} \right) \frac{\Omega_i}{k_j \omega_i} > 1
\]

and also \( \Omega_i/k_j \omega_i < v_p/v_e \), so that a rough criterion for instability is given as

\[
v_p/v_e > 5.
\]

To refine the stability criterion we return to Eq. (5) and look for a critical value of \( v_p \) which will lead to real frequency \( \omega \). The imaginary part of Eq. (5) then becomes
\[
\frac{\omega}{k_i v_i} - \frac{v_e}{v_i} + \frac{T_e}{T_i} \Gamma_i \frac{\omega}{k_i v_i} \exp \left\{ -\frac{1}{2} \left[ (\omega - \Omega_i) / k_i v_i \right]^2 \right\} = 0, \tag{9a}
\]
and the real part
\[
(T_e / T_i) \Gamma_i [\Omega_i / (\omega - \Omega_i)] = 1. \tag{9b}
\]
Substituting (9b) into (9a) we have
\[
\frac{v_e}{v_i} = \left( 1 + \frac{T_e}{T_i} \Gamma_i \right) \frac{\omega - \Omega_i}{v_i \Gamma_i T_i} + \frac{\omega - \Omega_i}{k_i v_i} \cdot \exp \left\{ -\frac{1}{2} \left[ (\omega - \Omega_i) / k_i v_i \right]^2 \right\}.
\]
The minimum comes for
\[
\frac{1}{2} \left( \frac{\omega - \Omega_i}{k_i v_i} \right)^2 \approx -\ln \frac{v_i}{v_i T_i} \approx \frac{1}{2} \ln \frac{m_i}{m_e}
\]
or
\[
\frac{\Omega_i}{k_i v_i} \approx \left( \ln \frac{m_i}{m_e} \right) \frac{T_i}{T_i + 1} \Gamma_i
\]
and the critical drift is then
\[
\frac{v_e}{v_i} \approx \left( \frac{T_i}{T_i + 1} \right) \left( \ln \frac{m_i}{m_e} \right) \approx 14 \frac{T_i}{T_i + 3}. \tag{10}
\]
This formula is not reliable for \( T_i \) much greater than \( T_i \), as then higher \( n \) values must be considered in Eq. (9b). Nonetheless one can see by inspection that as \( T_e / T_i \) decreases, the root moves closer to \( \Omega_i \) and the critical drift increases. Conversely as \( T_e / T_i \) increases, the critical drift decreases.

It would appear then that this instability near the ion-cyclotron frequency reduces the value of \( v_e / v_i \) necessary for instability by about an order of magnitude in the case of equal temperatures as compared to previous theories which consider only \( k_\perp = 0 \).

### III. NONLINEARIZED THEORY

From the derivation it is clear that the dispersion relation \( (\omega - \Omega_i) / \Omega_i = \Gamma_i (T_e / T_i) \) is insensitive to small changes in the distribution function while the growth rate depends critically on the distribution function. In particular, for a more general electron distribution, the growth rate \( \gamma \) is given by
\[
\gamma = \alpha \frac{\partial g}{\partial v_i} \bigg|_{v_i = \omega / k_i} \tag{11}
\]
where
\[
g = \int f N_+ dN_+ d\phi,
\]
\[
\alpha = 2 \Omega_i \Gamma_i (T_e / T_i) (\pi n_e^2 / n).
\]
In order to apply the nonlinear theory it is required that the growth rate in the linearized theory be proportional to \( \partial g / \partial v_i \) and \( \gamma_k / \omega_k \ll 1 \). In addition, the dispersion relation must be such that \( \omega_k + \omega_i \neq \omega_k - \gamma_k \). The ion-cyclotron instability considered in Sec. II satisfies all these necessary conditions.

In the nonlinear theory \( f_0(v) \) is replaced by a slowly varying function \( f(v, t) \) for which \( f(v, 0) = f_0(v) \), and \( f(v, t) \) varies in time according to
\[
\frac{\partial f(v, t)}{\partial t} = \left( \frac{e}{m} \right) \sum_k E_{-k} \cdot \nabla f_k.
\]
Inserting \( f_k(v) \) from Eq. (A12) of the Appendix we obtain
\[
\frac{\partial f}{\partial t} = \left( \frac{e}{m} \right)^2 \sum_k E_{-k} \cdot \nabla f
\]
\[
\cdot \left[ E_{+} \sin (\phi - \theta) \frac{\partial}{\partial v_x} + \frac{E_{+}}{(s + ik v_i)} \frac{\partial}{\partial v_x} \right] f(v, t)
\]
\[
= \left( \frac{e}{m} \right)^2 \sum_k \left[ E_{+} \cos (\phi - \theta) \frac{\partial}{\partial v_x} + E_{+} \frac{\partial}{\partial v_x} \right] f(v, t), \tag{13}
\]
where \( \theta \) is the azimuthal angle of \( k \) and
\[
s = s(k) = -i \left[ k_1 \right] \omega(k) + \gamma(k).
\]
Integrating over \( \phi \) we obtain
\[
\int \frac{\partial f}{\partial t} d\phi = \left( \frac{e}{m} \right)^2 \sum_k |E_{+}|^2 \frac{1}{s + ik v_i} \frac{\partial}{\partial v_i} \int f d\phi,
\]
and therefore,
\[
\frac{\partial g}{\partial t} = \left( \frac{e}{m} \right)^2 \sum_k |E_{+}(k)|^2 \frac{\partial}{\partial v_i} \left( \frac{1}{s + ik v_i} \frac{\partial g}{\partial v_i} \right). \tag{14}
\]
Replacing \( \sum_{\nu} \) by \( L / 2\pi \int dk_\perp \) we have, since \( |E_{+}(k)|^2 \) and the real part of \( 1 / (\omega_k + ik v_i) \) are even functions of \( k_\perp \) and the imaginary part of \( 1 / (\omega_k - ik v_i) \) is an odd function of \( k_\perp \)
\[
\left( \frac{e}{m} \right)^2 \frac{L}{2\pi} \int dk_\perp \frac{|E_{+}(k_\perp, k_i)|^2}{(\omega_k - ik v_i)}
\]
\[
= \left( \frac{e}{m} \right)^2 \frac{L}{2\pi v_i} \int_{k_i}^{\omega_k} dk_{\perp} |E_{+}(k_{\perp}, k_i)|^2 \frac{1}{k_{\perp} - \omega_k / v_i}
\]
\[
= \left( \frac{e}{m} \right)^2 \frac{L}{v_i} \left| E_{+}(k_{\perp}, \omega_k / v_i) \right|^2. \tag{15}
\]
Denoting \( E_{+}(k_{\perp}, \omega_k / v_i) \) by \( E_{+}(k_{\perp}, v_i) \) the equations of motion become
\[
\frac{\partial |E_{+}(k_{\perp}, v_i)|^2}{\partial t} = \alpha |E_{+}|^2 \frac{\partial g}{\partial v_i}, \tag{16}
\]
\begin{align}
\frac{\partial g(v_i, t)}{\partial t} &= \frac{\partial}{\partial v} \left( \frac{e}{m} \right)^2 L \sum_{k \perp} |E(k_\perp, v_i)|^2 \left( \frac{k_\parallel}{k} \right)^2 \frac{\partial g}{\partial v_i}, \\
&= g_0(v) \quad \text{for } v_i < 0, \quad v_i > v_i, \tag{17}
\end{align}

As discussed in Sec. II, \( \gamma \) has a broad maximum near \( k_\parallel = k_\perp \approx 1.5 \Delta \nu / v_i \), and this leads, after many e-folding times, to a very sharp resonance of \( |E(k_\perp, v_i)|^2 \) near \( k_\parallel = k_\perp \). Making use of this and

\begin{align}
\frac{k_\parallel}{k} &\approx \frac{k_\perp}{k} \approx \frac{v_i}{1.5 \Delta \nu},
\end{align}

we can sum Eqs. (16) and (17) over \( k_\perp \) to obtain

\begin{align}
\frac{\partial E(v_i)}{\partial t} &= \alpha_0 E(v_i) \frac{\partial g}{\partial v_i}, \\
\frac{\partial g}{\partial t} &= \frac{\partial}{\partial v} \beta E(v_i) \frac{\partial g}{\partial v}, \tag{19}
\end{align}

where

\begin{align}
E(v_i) &= \sum_{k \perp} |E(k_\perp, v_i)|^2, \\
\alpha_0 &= \alpha(k_\perp), \\
\beta &= \frac{1}{2.2} \frac{L}{L} \left( \frac{e}{m} \right)^2 \left( \frac{v_i}{v_\nu} \right)^2. \tag{20}
\end{align}

The temporal behavior of the unstable waves can be described as follows. At \( t = 0 \) the electric fluctuations are assumed to consist of random noise, and initially those waves with \( 0 \leq \omega_0 / k_\parallel < v_\nu \) grow with the linearized growth rate, and the spectrum is very peaked about the fastest growing waves. After \( \delta(k_\parallel, v_i) \) has grown to a sufficient amplitude the distribution function begins to diffuse according to Eq. (19). This tends to flatten the distribution function at the velocity corresponding to the fastest growing wave, and consequently to steepen the distribution function on either side of this point. This in turn increases the growth rate of those waves on either side of the fastest growing wave while decreasing the growth rate of the fastest growing wave. Thus as the distribution function diffuses the spectrum of waves spreads.

As discussed in reference 2, the asymptotic result is

\begin{align}
\lim_{t \to \infty} g(v_i, t) &= g_\infty \quad \text{for } 0 < v_i < v_i, \\
&= g_0(v) \quad \text{for } v_i < 0, \quad v_i > v_i, \tag{21}
\end{align}

where \( g_\infty \) is a constant and together with \( v_i \) is determined by

\begin{align}
\int_0^{v_i} [g_\infty - g_0(v)] dv = 0, \quad g_\infty = g_0(v_i), \tag{22}
\end{align}

and

\begin{align}
g_0(v_i) &= \int f_0(v) v_1 dv_1 dv_2. \tag{23}
\end{align}

The accompanying equilibrium spectrum of waves is given by

\begin{align}
\xi_\infty(v) &= \frac{\alpha}{\beta} \int_0^v [g_\infty - g_0(v')] dv'. \tag{24}
\end{align}

The asymptotic distribution function and the equilibrium wave spectrum are illustrated in Figs. 1 and 2.

\section*{IV. DIFFUSION COEFFICIENT}

The equilibrium spectrum of fluctuations gives rise to a diffusion of particles across the field lines and as will be shown this is a result of a resonance between particle and wave velocities of the type which leads to the velocity diffusion given by Eq. (19). This relation can be exploited to give a heuristic derivation of the diffusion coefficient.

It is well known that for sufficiently slowly varying fields the drift velocity across the field is

\begin{align}
\frac{dx}{dt} &= c \frac{E(t) \times \mathbf{B}}{B^2} = c \frac{E_\perp(t)}{B},
\end{align}

and

\begin{align}
D, t &= \langle (\Delta v)^2 \rangle = \left\langle \frac{c}{B} \left( \int_0^t E_\perp(t') dt' \right)^2 \right\rangle, \tag{24}
\end{align}

where

\begin{align}
\xi_\infty(v) &= \frac{\alpha}{\beta} \int_0^v [g_\infty - g_0(v')] dv'. \tag{25}
\end{align}

\begin{align}
\xi_\infty(v) &= \frac{\alpha}{\beta} \int_0^v [g_\infty - g_0(v')] dv'. \tag{26}
\end{align}

\begin{align}
\xi_\infty(v) &= \frac{\alpha}{\beta} \int_0^v [g_\infty - g_0(v')] dv'. \tag{27}
\end{align}

\begin{align}
\xi_\infty(v) &= \frac{\alpha}{\beta} \int_0^v [g_\infty - g_0(v')] dv'. \tag{28}
\end{align}
where $D_r$ is the spatial diffusion coefficient perpendicular to the field lines, and $\langle \rangle$ indicates an ensemble average. Similarly, the diffusion of particles in velocity space is determined by

$$\frac{dv_i}{dt} = (c/m)E_i(t)$$

and

$$D_r = \langle (\Delta v)^2 \rangle = \left( \frac{c}{m} \right)^2 \left( \int_0^t \langle E_i(t') \Delta v \rangle \, dt' \right)^2$$

(25)

Making use of the fact that $E_L/E_i = k_L/k_i \approx v/v_i$, we have

$$D_r = \frac{1}{t} \left( \frac{c}{B} \right)^2 \left( \int E_L E_i \, dt' \right)^2$$

$$= \frac{1}{t} \left( \frac{c}{B} \right)^2 \left( \frac{k_L}{k_i} \right)^2 \left( \int E_i \, dt' \right)^2$$

$$= \rho_i^2 \left( \frac{v}{v_i} \right) \frac{1}{v_i} \frac{\Delta v}{v_i} D_r.$$  

(26)

Now $D_r \approx (\Delta v)^2/\tau$ when we expect $D_r$ to be of the order of $v_L$ and $1/\tau$ to be of the order of $\gamma$. Thus $D_r = \rho_i^2 \gamma$ and (taking $v \approx v_D$)

$$D_r = \rho_i^2 \Omega_i (T_i/T) (v_D/v_i)^2.$$  

(27)

Since only those electrons with $0 < v < v_D$ can resonate with the plasma waves only these electrons diffuse. Since these represent only a fraction of the order of $v_D/v_i$, of the total number of electrons the average diffusion coefficient goes more nearly as $(v_D/v_i)^2$. Thus we are able to obtain the dependence of $D_r$ on the parameters in a simple way and we note that $D_r \propto \rho_i^2 \Omega_i = D_\gamma$, the Bohm diffusion coefficient.

A more rigorous derivation of this result which also determines the coefficient of proportionality is given below. For electric fields varying slowly compared to the electron cyclotron frequency the electrons drift across the magnetic field with a velocity $v_L = c E_L/B$ and thus

$$r_L = \frac{c}{B} \sum_k \int_0^t E_L(k) \exp(i k \cdot r_0)$$

$$\cdot \exp \left[ i \left( k_L v_L - \omega_L \right) t \right] \, dt$$

$$= \frac{c}{B} \sum_k E_L(k) \exp(i k \cdot r_0)$$

$$\cdot \exp \left[ i \left( k_L v_L - \omega_L \right) t - \frac{1}{i} \frac{v}{v_L} \right],$$  

(28)

and

$$\langle \Delta r_L \rangle^2 = \left( \frac{c}{B} \right)^2 \sum_k |E_L(k)|^2 \left\{ \sin \left[ \frac{1}{2} \left( k_L v_L - \omega_L \right) t \right] \right\}^2$$

(29)

where $\langle \rangle$ denotes an average over the initial phases and we have assumed that $\langle \sum_k E_k \Delta k \rangle = |E_k|^2$. For large $t$ the function

$$\sin \left[ \frac{1}{2} \left( k_L v_L - \omega_L \right) t \right] \left( k_L v_L - \omega_L \right)$$

is sharply peaked about $k_L v_L - \omega_L = 0$ and

$$\sum_k \rightarrow L/2\pi \int dk_L$$

can be evaluated by taking advantage of this to give

$$\langle \Delta r_L \rangle^2 = 2(c/B)^2 (L/v) \varepsilon(v) t, \quad D_r = 2(c/B)^2 (L/v) \varepsilon(v).$$  

(30)

We see here that only those terms for which $k_L v_L = \omega_L$ lead to time proportional diffusion and thus it is a resonance between particle and phase velocities which leads to diffusion.

Inserting $\varepsilon(v)$ from Eq. (23) we have

$$D_r = 8.9 \pi T_0 (T_i/T)^2 \left( \frac{v_D}{v_i} \right)^2 \int \left( \frac{g_\infty - g}{n} \right) dv.$$  

(31)

For $v_D \ll v$, 

$$\int \left( \frac{g_\infty - g}{n} \right) dv \approx A \left( \frac{v_D}{v_i} \right)^3$$

and

$$D_r = A \left( \frac{v_D}{v_i} \right)^2 \left( \frac{T_i}{T} \right)^2 \rho_i^2 \Omega_i,$$  

(33)

in agreement with our earlier results.

The diffusion of ions is more complicated and is not considered in the present paper. For a given physical situation, however, the ion diffusion and the need for charge neutrality must be included.

V. DISCUSSION

We have shown that this instability near the ion-cyclotron frequency reduces the critical current which can be drawn parallel to the field without producing instability by about an order of magnitude in the case of equal temperature as compared to previous theories which consider only $k_L = 0$. This appears to be the explanation of the results of D'Anjelo and co-workers who have observed instabilities in a cesium plasma and the critical current they observe is in agreement with results of Sec. II. In addition, the measured frequency of the unstable oscillations was $\omega \cong 1.2 \Omega_i$ also in agreement. Nonetheless, this explanation is not completely satisfactory since in the experiment $\Omega_\tau \approx 10$ where $\tau_\tau$ is the ion–ion collision...
time and in view of the low predicted growth rates it is difficult to see why collisional damping would not dominate.

The diffusion coefficient obtained is small compared to the “Bohm diffusion coefficient”, \( \rho_0 \Omega_i^2 \), although it has the same dependence on magnetic field, i.e., \( D \propto 1/B \). Indeed for the stellarator \( v_B/v_e \ll 1 \) while the experimental result is \( D \propto \rho_0^2 \Omega_i \), and it thus appears that “pumpout” is not due to this type of instability, at least in its simplest form. There is, however, the possibility that the particle collisions couple the external electric field (which produces the drift) to the fluctuations, leading to a much higher fluctuation amplitude. To answer this question, however, one must solve the nonlinear Fokker-Planck equation.

In the absence of such an external source of energy it seems unlikely that any microinstabilities for which \( \gamma/\omega \ll 1 \) will produce large macroscopic diffusion for the simple reason that the energy fed into electric fluctuations is small, i.e., or order \( \gamma/\omega \). For the case at hand, it is a fortiori small since, as remarked in Appendix A, only the fraction \( (kL_B)^2 \ll 1 \) of the energy given up by the electrons goes into electric field, the bulk of the energy going into ion kinetic energy.

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APPENDIX A

We consider an infinite homogeneous plasma with a magnetic field \( B \) in the direction of the \( z \) axis. As discussed by many authors, e.g., Bernstein, the perturbed electron distribution function is given in the linearized theory by

\[
f_s = \frac{e}{m} \sum_n J_n(\lambda_s) \exp(\imath n \phi - \lambda_s \sin \phi) \frac{E_1}{2} \left[ \frac{e^{\imath \phi}}{s + ik \nu_i + i(n+1) \Omega} \frac{\partial}{\partial \nu_i} + \frac{e^{-\imath \phi}}{s + ik \nu_i - i(n-1) \Omega} \frac{\partial}{\partial \nu_i} \right] f_0(\nu_i, \nu_i), \tag{A1}
\]

where \( J_n \) is the usual Bessel function, \( s = -\imath \omega \) and \( \lambda_s = (k \rho_i) \). For \( k \rho_i \ll 1, s \ll -\imath \omega \), we have \( \lambda_s \ll 1, s + ik \nu_i \ll \imath \omega \), for all but the fastest electrons and we can neglect all terms but \( n = 0 \) to obtain

\[
f_k = \frac{e}{m} \Omega_i \sin \phi \frac{\partial}{\partial \nu_i} \left[ \frac{E_1}{s + ik \nu_i} \frac{\partial g_0}{\partial \nu_i} \right] f_0(\nu_i, \nu_i). \tag{A2}
\]

The electronic charge density is given by

\[
\rho_e = -e \int d\nu f_k = -\frac{e}{m} E_1 \int_{-\omega}^{\omega} \frac{g_0}{s + ik \nu_i} \, d\nu_i \tag{A3}
\]

where \( g_0 = \int f_0(\nu) \, d\nu \).

Similarly the perturbed ion distribution function is given by

\[
F_k = -\frac{e}{M} \sum_{n=-\infty}^{\infty} J_n(-\lambda_i) \exp(\imath n \phi + i \lambda_i \sin \phi) \frac{E_1}{2} \left[ \frac{e^{\imath \phi}}{s + ik \nu_i - i(n + 1) \Omega}, \right. \frac{\partial}{\partial \nu_i} + \frac{e^{-\imath \phi}}{s + ik \nu_i - i(n-1) \Omega}, \frac{\partial}{\partial \nu_i} \right] F_0 \tag{A4}
\]

Taking \( F_0(\nu_i, \nu_i) \) to be a Maxwellian yields

\[
\rho_i = E_k \frac{\omega_B^2}{4 \pi} \sum_{n=-\infty}^{\infty} \Gamma_n [(k \rho_i)^2] \frac{i}{k \nu_i} \left\{ W \left( -\frac{i \nu_i + n \Omega}{k \nu_i} \right) + \frac{n \Omega}{i \nu_i - n \Omega} \left[ 1 + W \left( -\frac{i \nu_i + n \Omega}{k \nu_i} \right) \right] \right\}, \tag{A5}
\]

where \( \Gamma_n \) is defined in Sec. II.

If the particles have a drift velocity \( v_D \), we must replace \( s = -i \omega \), \( k \nu_i \) with \( \nu_B \). Thus using \( \nabla \cdot E = i k E_k \), and \( s = -i \omega \), we obtain the dispersion relation, Eq. (2).

It is worth noting that only a small fraction, \( (kL_B)^2 \ll 1 \), of the energy which is given up by those electrons in resonance with the waves, i.e., those electrons with \( k \nu_i = \omega/k \nu_i \), goes into electro-

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\(^\text{1}\) I. B. Bernstein, Phys. Rev. 109, 10 (1958).
static energy, and the bulk of the energy goes into the kinetic energy of wave motion. This is in contrast to the case of electron plasma oscillation for which the energy from resonance particles is evenly divided between kinetic and potential energy.\(^2\) To see this we note that the energy transfer is proportional to the "in-phase" part of the current \(j\) and that \(j_k = -(s/ik)\rho_k\). The rate of change of energy is thus

\[
\frac{\partial U}{\partial t} = -\frac{\partial}{\partial t} \sum_k \frac{|E_k|^2}{8\pi} = \sum_k E_k \cdot j_k
\]

\[
= - \sum_k |E_k|^2 \frac{s}{4\pi} \sum_i \left( \frac{\omega_{pi}}{k_i} \right)^2 \times \left\{ 1 + \left( \frac{\pi}{2} \right) \left( \frac{-\omega + i k_i v_i}{v_i} \right) + \Gamma_i \left( \frac{(k_i v_i)^2}{\Omega_i^2} \right) \right\}
\]

where

\[
U = \sum_i \frac{1}{2} m_i \int v^2 f_i(v) \, dv
\]
is the kinetic energy of the particles and we have taken only \(n = 0\) for electrons and \(n = 1\) for ions. For each \(k\) the real part is given by

\[
\frac{dU}{dt}_k = \frac{|E_k|^2}{4\pi (kL_{D_k})^2} \left\{ +\frac{\gamma}{\Gamma_i} \left( \frac{\pi}{2} \right) \left( \frac{-\omega + k_i v_i}{k_i v_i} \right) \right. \\
+ \frac{\Omega_i}{\omega - \Omega_i} \frac{T_i}{T_e} \right\}
\]

\[
= -\frac{|E_k|^2}{4\pi (kL_{D_k})^2} \left\{ \frac{\gamma}{\Omega_i T_e} \frac{T_i}{T_e} \left[ 1 + (kL_{D_k})^2 \right] \right. \\
+ \frac{\gamma}{\Gamma_i} \frac{T_i}{T_e} \left[ 1 + (kL_{D_k})^2 \right] \right\}
\]

(A7)

where we have evaluated \(\gamma\) and \(\omega\) from Eq. (2) without neglecting \((kL_{D_k})^2 \ll 1\).

\[
\left( \frac{\omega - \Omega_i}{\Omega_i} \right) = \frac{T_i}{T_e} \frac{\Gamma_i}{1 + (kL_{D_k})^2},
\]

\[
\gamma = \frac{T_i}{T_e} \frac{\Gamma_i \Omega_i}{1 + (kL_{D_k})^2} \left( \frac{\pi}{2} \right) \left( -\frac{\omega + k_i v_i}{k_i v_i} \right).
\]

(A8)

The first term in the curly brackets in Eq. (A7) comes from the bulk of the electrons and represents the energy fed into the electron kinetic energy of the waves. The second term comes from the resonant electrons and represents the energy drawn from these particles. The third term is the energy fed into the ion kinetic energy of the wave. Note that the terms from the resonant electrons and from the ions are both larger by a factor of \(1/\Gamma_i\) than the electron kinetic energy term and their difference is \(-\gamma[1 + (kL_{D_k})^2]\). Thus the total change in particle energy is just

\[
\frac{dU}{dt} = -\gamma \frac{|E_k|^2}{4\pi} = -\frac{\partial}{\partial t} \frac{|E_k|^2}{8\pi}.
\]

(A9)

We may describe this as follows. The resonant electrons give up energy to the waves at a rate of order \(\gamma/\Gamma_i (kL_{D_k})^2\). The waves in turn feed energy into the ions at a rate of \(\gamma/\Gamma_i (kL_{D_k})^2\) and into the bulk of the electrons at a rate of order \(\gamma/\Gamma_i \Gamma_i (kL_{D_k})^2\). The net rate of electrostatic energy change is, however, of order \(\gamma \ll \gamma/\Gamma_i (kL_{D_k})^2\). Thus the ion-cyclotron waves have only the fraction \((kL_{D_k})^2 \ll 1\) of their energy as potential energy.