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Second-order dust acoustic wave theory

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Abstract

A second-order perturbation theory for non-dispersive, undamped dust acoustic waves is presented. The analysis leads to a second-order wave equation with source terms consisting of (nonlinear) products of first-order terms. The nonlinear effects included in this analysis might be useful in explaining the non-sinusoidal waveforms that are observed with large-amplitude, self-excited dust acoustic waves.

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Dust acoustic waves are low-frequency, longitudinal dust density waves that propagate through a fluid of charged dust particles suspended in a plasma. The linear theory (first-order perturbations) of small-amplitude dust acoustic waves, including dispersive effects, was first derived by Rao, Shukla and Yu (RSY) [1]. For finite-amplitude waves, RSY derived a generalized Boussinesq equation of third order, and showed that under certain conditions, this equation reduced to a generalized Korteweg–de Vries (KdV) equation, which they solved using the standard reductive perturbation method that admitted localized solutions. The above-mentioned pioneering paper of RSY has motivated a considerable amount of further work on nonlinear dust acoustic waves and, in particular, dust acoustic solitary waves (see, e.g., [2]).

This paper presents a very simplified second-order wave theory, with the motivation for capturing the basic physics of finite-amplitude dust acoustic waves. The emphasis here will focus on the minimum physics necessary to interpret the non-sinusoidal dust acoustic waveforms that are typically observed in experiments (see, e.g., [3]). We adopt the simplest fluid description with the following assumptions: (i) the dusty plasma is uniform and homogeneous; (ii) the dust is treated as a cold fluid, $T_\mathrm{d} = 0$; (iii) the dust charge is constant; (iv) charge neutrality is assumed; (v) there are no dissipation mechanisms present; and (vi) the waves are planar (one-dimensional). Under these conditions, the system is described by the (nonlinear) continuity and momentum equations

$$\frac{\partial n_\mathrm{d}}{\partial t} + u_\mathrm{d}\frac{\partial n_\mathrm{d}}{\partial x} + n_\mathrm{d}\frac{\partial u_\mathrm{d}}{\partial x} = 0, \qquad (1a)$$

$$\frac{\partial u_\mathrm{d}}{\partial t} + u_\mathrm{d}\frac{\partial u_\mathrm{d}}{\partial x} - \frac{eZ_\mathrm{d}}{m_\mathrm{d}}\frac{\partial \varphi}{\partial x} = 0, \qquad (1b)$$

where n_d is the dust density, u_d is the dust fluid velocity, $eZ_\mathrm{d} = Q_\mathrm{d}$ is the dust charge (taken to be negative), m_d is the dust mass and φ is the electric potential. Since the phase speed of the dust acoustic wave is well below the electron and ion thermal speeds, the electrons and ions are taken to be in Boltzmann equilibrium,

$$n_\mathrm{e} = n_\mathrm{e0}\exp(e\varphi/kT_\mathrm{e}), \qquad (2a)$$

$$n_\mathrm{i} = n_\mathrm{i0}\exp(-e\varphi/kT_\mathrm{i}), \qquad (2b)$$

where $n_{e(\mathrm{i})}$ is the electron (ion) density, and $T_{e(\mathrm{i})}$ is the electron (ion) temperature. The model is completed by taking $\partial^2\varphi/\partial x^2 = 0$, which amounts to neglecting dispersive effects, and implies the neutrality condition

$$n_\mathrm{i} = n_\mathrm{e} + Z_\mathrm{d} n_\mathrm{d}, \qquad (3)$$

which, in the zero-order state, reads $n_\mathrm{i0} = n_\mathrm{e0} + Z_\mathrm{d} n_\mathrm{d0}$.

Let ψ represent any of the variables (n_e, n_i, n_d, u_d, φ), and expand ψ in a perturbation series with a small parameter ε as $\psi = \psi_0 + \varepsilon\psi_1 + \varepsilon^2\psi_2$, with $u_\mathrm{d0} = 0$, and $\varphi_0 = 0$. The Boltzmann relations (equation (2)) are expanded to second order in φ and when combined with the neutrality condition allows us to write the first- and second-order wave potentials in terms of the first- and second-order dust densities as

$$\varphi_1 = -A_1 n_\mathrm{d1}, \qquad (4a)$$

$$\varphi_2 = A_2 n_\mathrm{d1}^2 - A_1 n_\mathrm{d2}, \qquad (4b)$$

where

$$A_1 = eZ_\mathrm{d}\lambda_\mathrm{D}^2/\varepsilon_\mathrm{o}, \quad A_2 = e^3 Z_\mathrm{d}^2\lambda_\mathrm{D}^6\left(1-\alpha\tau^2\right)\big/\left(2\varepsilon_\mathrm{o}^2 kT_\mathrm{i}\lambda_\mathrm{Di}^2\right),$$

$$\lambda_\mathrm{D} = \lambda_\mathrm{De}\lambda_\mathrm{Di}\Big/\sqrt{\lambda_\mathrm{De}^2+\lambda_\mathrm{Di}^2}, \quad \lambda_{\mathrm{D}j} = \left(\varepsilon_\mathrm{o} kT_j/e^2 n_{j0}\right)^{1/2},$$

$$j = (\mathrm{i}, \mathrm{e}), \quad \alpha = n_\mathrm{e0}/n_\mathrm{i0},$$

and $\tau = T_i/T_e$. Equations (1*a*) and (1*b*) are expanded to second order in ε^2 and terms having like powers of ε are equated to provide the first- and second-order continuity equations

$$\frac{\partial n_{d1}}{\partial t} + n_{d0}\frac{\partial u_{d1}}{\partial x} = 0, \quad (5a)$$

$$\frac{\partial n_{d2}}{\partial t} + n_{d0}\frac{\partial u_{d2}}{\partial x} + \frac{\partial (n_{d1}u_{d1})}{\partial x} = 0, \quad (5b)$$

and the first- and second-order momentum equations

$$\frac{\partial u_{d1}}{\partial t} - \frac{eZ_d}{m_d}\frac{\partial \varphi_1}{\partial x} = 0, \quad (6a)$$

$$\frac{\partial u_{d2}}{\partial t} + u_{d1}\frac{\partial u_{d1}}{\partial x} - \frac{eZ_d}{m_d}\frac{\partial \varphi_2}{\partial x} = 0. \quad (6b)$$

Equation (4*b*) is used in (6*b*) to express φ_2 in terms of n_{d1} and n_{d2} in equation (6*b*). To obtain an equation for n_{d2}, u_{d2} must be eliminated in (5*b*) and (6*b*) by taking $\partial/\partial t$ on (5*b*) and $\partial/\partial x$ on (6*b*) and equating the mixed partial derivatives of u_{d2}. Collecting the second-order terms on one side of the equation and first-order terms on the other side, we obtain the following second-order wave equation for n_{d2}:

$$\frac{\partial^2 n_{d2}}{\partial x^2} - \frac{1}{C_{da}^2}\frac{\partial^2 n_{d2}}{\partial t^2} = \frac{A_2}{A_1}\frac{\partial^2 n_{d1}^2}{\partial x^2} - \frac{n_{d0}}{2C_{da}^2}\frac{\partial^2 u_{d1}^2}{\partial x^2} + \frac{1}{C_{da}^2}\frac{\partial^2 (n_{d1}u_{d1})}{\partial x \partial t}, \quad (7)$$

where

$$C_{da} = \lambda_D \omega_{pd}, \quad \omega_{pd} = \sqrt{e^2 Z_d^2 n_{d0}/\varepsilon_o m_d}.$$

Equation (7) can be simplified since the solutions to the first-order equations are assumed to be known, i.e.

$$\{n_{d1}, u_{d1}\} = \{\tilde{n}_{d1}, \tilde{u}_{d1}\}\exp[i(kx - \omega t)],$$

where the quantities with tildes are the amplitudes, and k and ω are the wavenumber and angular wave frequency, respectively. Then from (5*a*) we have $u_{d1} = (\omega/k)(n_{d1}/n_{d0})$, so that (7) can be written as

$$\frac{\partial^2 n_{d2}}{\partial x^2} - \frac{1}{C_{da}^2}\frac{\partial^2 n_{d2}}{\partial t^2} = A\frac{\partial^2 n_{d1}^2}{\partial x^2} + B\frac{\partial^2 n_{d1}^2}{\partial x \partial t}, \quad (8)$$

where

$$A = A_2/A_1 - \omega^2/\left(2k^2 n_{d0} C_{da}^2\right)$$

and

$$B = \omega/\left(k n_{d0} C_{da}^2\right).$$

Equation (8) is the second-order equation for dust acoustic waves, which is similar in form to the second-order equation for sound waves [4], in which the first-order terms on the rhs appear as *source terms* for the second-order wave equation. The source terms on the rhs of (8) are products of the (known) first-order terms, and thus give rise to second harmonic terms $\exp[2i(kx - \omega t)]$ in the inhomogeneous wave equation. It is clear that carrying out the perturbation analysis

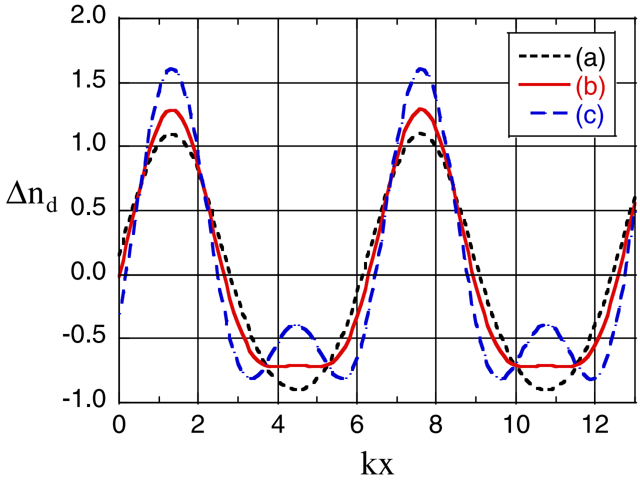

Figure 1. Non-sinusoidal (asymmetric) waveforms obtained from second-order dust acoustic wave theory (equation (8)) for $t = 0$, $\vartheta = -1.33$, $k = 1\ \mathrm{m}^{-1}$, $\tilde{n}_{d1} = 1$, with (a) $\tilde{n}_{d2} = 0.10$, (b) $\tilde{n}_{d2} = 0.29$ and (c) $\tilde{n}_{d2} = 0.61$.

to second order not only leads to quantitative corrections to the first-order quantities, but also adds qualitatively new effects that are not contained in a linear wave analysis. Dropping the expansion parameter ε, the dust density to second order can be written as $n_d = n_{d0} + n_{d1} + n_{d2}$, or the excess dust density as $\Delta n_d \equiv n_d - n_{d0} = n_{d1} + n_{d2}$. Since the second-order term involves the second harmonics, the wave solution for $\Delta n_d(x, t)$ can then be written to second order as

$$\Delta n_d(x, t) = \tilde{n}_{d1}\cos(kx - \omega t + \vartheta) + \tilde{n}_{d2}\cos[2(kx - \omega t + \vartheta)], \quad (9)$$

where $\tilde{n}_{d1}$ and $\tilde{n}_{d2}$ are the first- and second-order amplitudes and ϑ is an arbitrary phase factor. Figure 1 shows a plot of (8) for $t = 0$, $\vartheta = -1.33$, $k = 1\ \mathrm{m}^{-1}$ and $\tilde{n}_{d1} = 1$, for three values of $\tilde{n}_{d2}$, (a) 0.10, (b) 0.29 and (c) 0.61. Even the relatively small second-order term in case (a) produces a non-symmetric waveform. The effect of the second-order term for case (b) is to make the wave crests sharper and the wave troughs flatter. Such waveforms are typically observed in laboratory dusty plasma experiments for self-excited dust acoustic waves [3].

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References

[1] Rao N N, Shukla P K and Yu M Y 1990 *Planet. Space Sci.* **38** 543–6
[2] Shukla P K and Mamun A A 2002 *Introduction to Dusty Plasma Physics* (Bristol: Institute of Physics Publishing) pp 195–227
[3] Teng L-W, Chang M-C, Tseng Y-P and Lin I 2009 *Phys. Rev. Lett.* **103** 245005
Heinrich J R, Kim S-H, Meyer J K and Merlino R L 2011 *Phys. Plasmas* **18** 113706
Merlino R L, Heinrich J R, Kim S-H and Meyer J K 2012 Nonlinear dust acoustic waves and shocks *Phys. Plasmas* accepted
[4] Morse P M and Uno Ingard P M 1968 *Theoretical Acoustics* (New York: McGraw-Hill) p 866