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TWO STREAM INSTABILITY AT THE ION GYRO FREQUENCY

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In a recent experiment, d'Angelo and coworkers have observed oscillation near the ion cyclotron frequency in a tepid Cs plasma when the electron drift velocity parallel to the magnetic field exceeds about three times the ion thermal velocity. In the condition of his experiment the ratio of kinetic to magnetic pressure is $\beta \approx 10^{-7}$ and $T_e \cong T_i$. The usual theory of the two stream instability, which considers only waves propagating parallel to the magnetic field, would predict stability until the electron drift velocity becomes comparable to the electron thermal velocity. Only in situations where $T_e \gtrsim 10\ T_i$ does the critical velocity approach the ion thermal velocity. In the following we point out that for a collisionless plasma instability occurs at a much lower velocity for electrostatic waves near the ion cyclotron frequency and propagating at large angles to the field.

We consider a homogeneous infinite plasma in which the ions and electrons each have a Maxwellian distribution at a characteristic temperature with the center of the Maxwellians displaced by a drift velocity u. We will work in the frame where the ions are at rest. If u = 0 the plasma is evidently stable. We would therefore expect that instability could only occur if $k_{||} u > \nu$ where ν is the wave frequency and $k_{||}$ the wave number parallel to the field. This is the condition that the peak of electron distribution be moving slightly faster than the wave--the usual condition for being able to put energy into the wave.

In this case if we put $\nu \sim \Omega_i$, the ion cyclotron frequency, and $u \sim v_{thi}$--the case we wish to discuss--we obtain the condition $k_{||} R_{Li} > 1$. The significance of this is that for large k and low β only pure electrostatic waves are possible. Thus if we write down the dispersion matrix $\nabla \times \nabla \times \vec{E} + \ddot{\vec{E}} = -4\pi \dot{\vec{j}}$ in a coordinate system in which one of the axes is parallel to $\vec{k}$, determining $\vec{j}$ from the Boltzmann equation, we find the dispersion matrix has the following structure:

$$\begin{pmatrix} k^2-\nu^2+(\alpha_{11}) & (\alpha_{12}) & (\alpha_{13}) \\ (\alpha_{21}) & k^2-\nu^2+(\alpha_{22}) & (\alpha_{23}) \\ (\alpha_{31}) & (\alpha_{32}) & -\nu^2+(\alpha_{33}) \end{pmatrix} = 0 \qquad (1)$$

where the quantities (α_{ij}) arise from the plasma currents and are all of order $\frac{\nu^2}{(k\lambda_D)^2}$. We note that for $k^2 >> \nu^2$, $\frac{\nu^2}{(k\lambda_D)^2}$. The only possible root is $\nu^2 = (\alpha_{33})$--the pure electrostatic mode in which $\vec{E} \parallel \vec{k}$. If we put $\nu \sim \Omega_i$, $k \; \frac{\Omega_i}{v_{th_i}}$ then $(k\lambda_D)^2 \approx \frac{B^2}{4\pi n m_i c^2} << 1$ and $\frac{k^2(k^2\lambda_D^2)}{\nu^2} = \frac{B^2}{4\pi n m_i v_{th_i}^2} = \frac{1}{\beta} >> 1$. Thus we need only consider pure electrostatic modes.

For this case the dispersion relation has been given by many authors, e.g., Bernstein, Phys. Rev. 109, 10 (1958)

$$\sum_{j=i,\ell} \sum_n \frac{1}{T_j} \Gamma_n (k_\perp^2 R_{L_j}^2) \left\{ W\left(\frac{\nu+k_{\parallel} u_j + n\Omega_j}{|k_{\parallel}| \, v_{th_j}}\right) - \frac{n\Omega_j}{\nu+k_{\parallel}u_j+n\Omega_j}\left[1 + W\left(\frac{\nu+k_{\parallel}u_j+n\Omega_j}{|k_{\parallel}| \, v_{th_j}}\right)\right]\right\} \qquad (2)$$

Here T is temperature, u_j the drift velocity of the j species, $\Gamma_n(x) = e^{-x} I_n(x)$--$I_n$ is the usual Bessel function of imaginary argument, and

$$W(x) = -1 + \frac{x}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-y^2}}{x+y} \, dy. \qquad (3)$$

The contour of the integral may be taken along the real axis for x in the lower half plane (growing waves). Limiting forms are given by

$$W \approx -1+i\sqrt{\pi}\,x \qquad |x| << 1$$

$$W \approx \frac{1}{2x^2} \qquad |x| >> 1. \tag{4}$$

For all real x, $IP(W) = i\sqrt{\pi}\,xe^{-x^2}$.

The limiting form given for large x is not valid for highly damped waves which are of no concern here.

Since we are concerned with wave lengths comparable to the ion-gyro-radius $k_\perp R_{L_e} << 1$; $\Gamma_{0,e} = 1$; $\Gamma_{n\neq 0,e} = 0$. Moreover we see that since $\frac{u}{v_{th_e}} << 1$ the argument of the electron W function is very small. Since we are concerned with a wave nearly at resonance with the ion-gyrofrequency we may neglect all the ion terms except n = -1. This gives us

$$0 = \left(-1+i\sqrt{\pi}\,\frac{(\nu+k_{||}u)}{|k_{||}|\,v_{th_e}}\right) + \frac{T_e}{T_i}\,\Gamma_1\left[W\left(\frac{\nu-\Omega_i}{|k_{||}|\,v_{th_i}}\right) + \frac{\Omega_i}{\nu-\Omega_i}\left\{1+W\left(\frac{\nu-\Omega_i}{|k_{||}|\,v_{th_i}}\right)\right\}\right] \tag{5}$$

We may obtain an approximate solution by noting that a condition for solution is a large argument for the ion W function as otherwise the large imaginary part (ion cyclotron damping) will give a damped solution.

If

$$\left|\frac{\nu-\Omega_i}{k_{||}v_{th_i}}\right| >> 1 \tag{6}$$

we have simply

$$\frac{\Omega_i}{\nu-\Omega_i}\,\frac{T_e}{T_i}\,\Gamma_1 = \left[1-i\sqrt{\pi}\left(\frac{\Omega_i-k_{||}u}{k_{||}v_{th_e}}\right)\right]$$

$$\nu-\Omega_i = \Omega_i\,\frac{T_e}{T_i}\,\Gamma_1\left\{1+i\sqrt{\pi}\left(\frac{\Omega_i}{k_{||}v_{th_e}} - \frac{u}{v_{th_e}}\right)\right\}. \tag{7}$$

Here we have chosen $k_{\parallel}$ negative as the direction of propagation for instability and used $\nu \sim \Omega_i$.

We note that Γ_1 has a very flat maximum at $k_\perp^2 R_L^2 \approx 1.5$ attaining there a value .22.

We conclude therefore that the maximum growth rate is given by

$$-IP(\nu) \approx .4 \frac{T_e}{T_i} \Omega_i \frac{u}{v_{th_e}} \tag{8}$$

occuring for $k_\perp^2 R_L^2 \sim 1$ and $k_{\parallel} > \frac{\Omega_i}{u}$. ← $\lambda_{\parallel} < \sim 15\,\text{cm}$

Moreover from Eq. (6) we must have

$$\frac{\nu - \Omega_i}{k_{\parallel} v_{th_i}} \approx .2 \frac{T_e}{T_i} \frac{\Omega_i}{k_{\parallel} v_{th_i}} > 1 \qquad \lambda_{\parallel} > 6\,\text{cm}$$

and also $\frac{\Omega}{k_{\parallel} v_{th_e}} < \frac{u}{v_{th_e}}$ so that a rough criterion for instability is given as

$$\frac{u}{v_{th_i}} > 5.$$

To refine the stability criterion we return to Eq. (5) and look for a critical value of u which will lead to real frequency ν. The imaginary part of Eq. (5) then becomes

$$\frac{\nu}{k_{\parallel} v_{th_e}} - \frac{u}{v_{th_e}} + \frac{T_e}{T_i} \Gamma_1 \frac{\nu}{k_{\parallel} v_{th_i}} e^{-\left(\frac{\nu - \Omega_i}{k_{\parallel} v_{th_i}}\right)^2} = 0 \tag{9a}$$

and the real part

$$\frac{T_e}{T_i} \Gamma_1 \frac{\Omega_i}{\nu - \Omega_i} = 1. \tag{9b}$$

~ 1 $\qquad \nu = 2\Omega_i$

We have neglected the small real part of W here as its argument is large. Substituting (9b) into (9a) we have

$$\frac{u}{v_{th_e}} = \left(1 + \frac{T_e}{T_i}\Gamma_1\right)\left[\frac{\nu-\Omega_i}{k_{\parallel} v_{th_i}}\,\frac{v_{th_i}}{v_{th_e}}\,\frac{T_i}{\Gamma_1 T_e} + \frac{\nu-\Omega_i}{k_{\parallel} v_{th_i}}\, e^{-\left(\frac{\nu-\Omega_i}{k_{\parallel} v_{th_i}}\right)^2}\right]$$

The minimum comes for

$$\left(\frac{\nu-\Omega}{k_{\parallel} v_{th_i}}\right)^2 \approx -\ln\frac{v_{th_i}}{v_{th_e}}\,\frac{T_1}{\Gamma_1 T_e} \approx \frac{1}{2}\ln\frac{m_i}{m_e}$$

or

$$\frac{\Omega}{k_{\parallel} v_{th_i}} \approx \sqrt{\frac{1}{2}\ln\frac{m_i}{m_e}}\;\frac{T_i}{T_e\Gamma_i},$$

? $\lambda_{\parallel} \approx 1$ cm

and the critical drift is then

$$\frac{u}{v_{th_i}} \approx \left(\frac{T_i}{\Gamma_1 T_e} + 1\right)\sqrt{\frac{1}{2}\ln\frac{m_i}{m_e}} \approx 10\,\frac{T_i}{T_e} + 2 \qquad (10)$$

This formula is not reliable for $T_e > T_i$ as then higher n values must be considered in Eq. (9b). Nonetheless one can see by inspection that as T_e/T_i increases the root moves closer to Ω_i and the critical drift decreases.

It would appear then that this instability near the ion cyclotron frequency reduces the amount of current which can be drawn parallel to the field by about an order of magnitude in the case of equal temperature as compared to previous theories which consider only $k_{\perp} = 0$. Nonetheless it appears far from a satisfactory explanation of d'Angelo's results for the following reasons:

Exp. $\frac{v_e \text{ drift } v_i}{v_{i\,m}\ th} \approx 11$

(1) The predicted critical velocity is still about a factor 4 too high as compared with his results. Whether the experimental error is this large we cannot say (for example whether for some reason T_e is a few times larger than T_i). Moreover we predict oscillations at about $1.2\,\Omega_i$ rather than at Ω_i. O.K.

(2) In practise one would not expect to have a double Maxwellian distribution. Rather the electron distribution function near v = 0 should be much flatter since Coulomb collisions are more frequent for low energy particles. This should increase the critical velocity somewhat. (O.K.)

(3) This theory is based on the collisionless equations. In d'Angelo's experiments $\Omega_i \tau_{coll} \sim 10$, where τ_{coll} is the ion-ion collision time. In view of the low growth rates we calculate it is difficult to see why collisional damping would not dominate or in fact how any coherent wave near the ion cyclotron frequency can exist.