General Astronomy (29:61)
Fall 2012
Lecture 7 Notes, September 10, 2012

1 Summary of Chapter 2

There are a number of items from Chapter 2 that you should be sure to understand.

1.1 Terminology

A number of technical terms arise in this chapter for important concepts. Be sure you understand what they mean.

- opposition
- inferior conjunction
- superior conjunction
- quadrature
- elongation
- Be sure you understand how do determine the distance (in astronomical units) of an inferior planet and a superior planet.

1.2 Kepler's Laws of Planetary Motion

Be sure you not only know what they are, but also how you can carry out computations using them. Know the critical elements of an ellipse.

2 Kepler's Laws from Newtonian Mechanics

The point of Chapter 3 is to demonstrate how one can understand Kepler's Laws of Planetary Motion from very fundamental laws of physics, Newton's Laws of Motion.

2.1 Vectors

Vectors are the mathematical objects we use to discuss the motion of celestial objects. Vectors have magnitude *and* direction. We write them as

$$\vec{v} = v\hat{e} \tag{1}$$

where $\hat{e}$ is a *unit vector* pointing in the direction of the vector. You can also write vectors in terms of their components in the different directions of a coordinate system (like a Cartesian coordinate system).

$$\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k} \tag{2}$$

where $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ are unit vectors in the x, y, and z directions. **Look at Figures 3.1 and 3.2** for an illustration of unit vectors.

2.2 Vector Algebra

Just like we can add, subtract, and multiply numbers, we can add, subtract, and multiply vectors.

In high school, you learned how to add two vectors, $\vec{A}$ and $\vec{B}$.

$$\vec{C} = \vec{A} + \vec{B} \tag{3}$$

You add the vectors head to tail.

2.3 Vector Multiplication

You can also multiply vectors together. There are two kinds of vector multiplication. The *dot product* yields a number, not a vector.

$$\vec{A} \cdot \vec{B} = AB\cos\theta \tag{4}$$

where θ is the angle between the vectors. The *cross product* of two vectors yields another vector

$$\vec{C} = \vec{A} \times \vec{B} = AB\sin\theta\hat{k} \tag{5}$$

where again θ is the angle between the two vectors, and $\hat{k}$ is a unit vector perpendicular to the plane containing $\vec{A}$ and $\vec{B}$. The *right hand rule* tells you the direction of the cross product, given $\vec{A}$ and $\vec{B}$. Both the dot product and the cross product are very important, and find applications in physics.

3 Newton's Laws of Motion

Newton's Laws of Motion are the basis of the field of physics called classical mechanics. Look on page 61 for their definition.

1. An object's velocity ($\vec{v}$, a vector) remains constant unless a net outside force acts upon it.

2. If a net outside force ($\vec{F}$, a vector)acts on an object, its acceleration is directly proportional to the force and inversely proportional to the mass of the object. In short,
$$\vec{F} = m\vec{a} \tag{6}$$
where $\vec{F}$ is the outside force, m is the mass, and $\vec{a}$ is the acceleration (a vector).

3. Forces come in pairs, equal in magnitude and opposite in direction.

Acceleration is the time derivative of the velocity, $\vec{a} = \frac{d\vec{v}}{dt}$

4 Newton's Expression for the Gravitational Force

Consider a mass M, another mass m, separated by a vector $\vec{r}$ that points from M and m, $\vec{r} = r\hat{r}$. The force exerted on m by M is then given by

$$\vec{F} = -\frac{GMm}{r^2}\hat{r} \tag{7}$$

A force that acts along the vector separating the two objects is called a *central force*.

In this course, we will restrict ourselves to cases where $M \gg m$ (think of the Sun and planets), but the general case $M \simeq m$ is just as easy.

5 Derivation of Kepler's Laws from Newton's Laws

5.1 Kepler's 2nd Law

Kepler's 2nd Law is the easiest to derive, and so we deal with that one first (Kepler obviously didn't know this when he numbered his laws), and it interesting to understand that it is a consequence of the conservation of angular momentum.

Angular momentum is one of the most important concepts in physics. For an object moving with respect to a coordinate system (think of a system whose origin is the center of the Sun),

$$\vec{L} = m\vec{r} \times \vec{v} = \vec{r} \times (m\vec{v}) = \vec{r} \times \vec{p} \tag{8}$$

The linear momentum is given by $\vec{p} = m\vec{v}$.

If we let θ be the angle between $\vec{r}$ and $\vec{r}$, then

$$L = mvr \sin\theta \tag{9}$$

If force is applied to an object, it can change the angular momentum.

$$\frac{d\vec{L}}{dt} = \vec{T} = \vec{r} \times \vec{F} \tag{10}$$

where $\vec{T}$ is the *torque*. If $\vec{T} = 0$, the angular momentum is constant, or conserved.

Let's see what the torque is for a central force.

$$\vec{T} = \vec{r} \times \vec{F} \tag{11}$$

$$\vec{T} = \vec{r} \times \left(\frac{GMm}{r^2}\right)\hat{r} \tag{12}$$

$$\vec{T} = \left(\frac{GMm}{r^2}\right)\vec{r} \times \hat{r} \tag{13}$$

Since $\vec{r} \times \hat{r} = 0$, there is no torque exerted by a central force, and the angular momentum is constant.

Let's see what the consequences are of conservation of angular momentum. **Look at Figure 3.3**. In a time interval dt the object of mass m moves, and sweeps out an area dA. We use the result from geometry that the area of a triangle is

$$dA = \frac{1}{2}bh \tag{14}$$

where $b = r$ and $h = v\sin\theta dt$, θ being the angle between $\vec{r}$ and $\vec{v}$.

So,

$$dA = \frac{1}{2}bh = \frac{1}{2}rv\sin\theta dt \tag{15}$$

And

$$\frac{dA}{dt} = \frac{1}{2}rv\sin\theta = \frac{L}{2m} \tag{16}$$

So $\frac{dA}{dt} = C$, a constant. This is Kepler's 2nd Law. A nice illustration of Kepler's 2nd Law is given in **Figure 2.18** of the book.

5.2 Kepler's 1st Law

Newton's Laws can be used to show that the equation for an orbit, given by $r(\theta)$, is described by Equation 3.34.

$$r(\theta) = \frac{L^2}{GMm^2(1 + e\cos\theta)} \tag{17}$$

The coordinates (r, θ) are *polar coordinates* centered on one of the focuses of the ellipse. The parameter e is very important, and is called the *eccentricity*. Its significance will be described shortly.

I think the derivation is too advanced for this course, but you can look at the discussion on pp66 and 67 for yourself.

This is the equation for geometric figures called *conic sections*, and include an ellipse, a parabola, or a hyperbola. Check out **Figure 3.5**.

Ellipses occur in the case of $0 \leq e < 1$. The geometry of an ellipse is shown in **Figure 3.6**. Be sure you known the following terms which are illustrated in this picture, or discussed in the text.

- major axis
- minor axis
- eccentricity
- focus
- true anomaly

5.2.1 Mean Size of an Orbit

While an ellipse is a precise description of an orbit, sometimes we want to know the mean (or average) distance of the object m from M. The astronomical unit is defined as the mean distance between the Sun and Earth. Let's work out what this is for an ellipse.

Look at **Figure 3.7**. Imagine putting dots along the orbit, and measuring the distance between the focus to each dot, and averaging the values. What is this number?

Let's imagine organizing these points into sets of two points, one at (x, y) and the other at $(-x, y)$, as shown in **Figure 3.8**. Let r be the distance from the right-hand focus to the first point, and $r^{'}$ the distance to the second point. The average distance of these two points is then $\frac{1}{2}(r + r^{'})$.

We could also choose the other, left-hand focus to make our measurements. In this case, the average distance of the two points from this second focus is $\frac{1}{2}(\rho + \rho^{'})$

Now, one of the definitions of an ellipse is that it is the set of points in a plane figure such that the sum of the distances from the point to the two focuses is the same number, say A. We then have that

$$r + \rho = A = r^{'} + \rho^{'} \tag{18}$$

If you look at **Figure 3.8** you can probably satisfy yourself that $r^{'} = \rho$, so we have

$$r + r^{'} = A \tag{19}$$

It only remains to determine the constant A. Since this constant applies to every point on the ellipse, we can choose any one we want to determine A. We can make life easy for ourselves by choosing the "perihelion" point on the x axis on the right side. You can easily show that $A = 2a$, so

$$\frac{1}{2}(r + r^{'}) = a \tag{20}$$

Notice that the mean distance doesn't depend on x or y. If it doesn't depend on (x,y) for these two points, it won't for an of the other points on the ellipse either, so we can conclude that the average distance for all points on the ellipse is a. The average distance of the object m from M is a.