General Astronomy (29:61)
Fall 2012
Lecture 8 Notes, September 12, 2012

1 Newton's Expression for the Gravitational Force

Consider a mass M, another mass m, separated by a vector $\vec{r}$ that points from M and m, $\vec{r} = r\hat{r}$. The force exerted on m by M is then given by

$$\vec{F} = -\frac{GMm}{r^2}\hat{r} \tag{1}$$

A force that acts along the vector separating the two objects is called a *central force*.

In this course, we will restrict ourselves to cases where $M \gg m$ (think of the Sun and planets), but the general case $M \simeq m$ is just as easy.

The constant G is the gravitational constant. It is one of the *fundamental constants of physics*, and is an extremely important number. It tells us how strong the force of Gravity is. It has a value of $G = 6.6726 \times 10^{-11}$ in SI units. The units of G are $\mathrm{m}^3\mathrm{s}^{-2}\mathrm{kg}^{-1}$.

2 Derivation of Kepler's Laws from Newton's Laws

2.1 Kepler's 2nd Law

Kepler's 2nd Law is the easiest to derive, and so we deal with that one first (Kepler obviously didn't know this when he numbered his laws), and it interesting to understand that it is a consequence of the conservation of angular momentum.

Angular momentum is one of the most important concepts in physics. For an object moving with respect to a coordinate system (think of a system whose origin is the center of the Sun),

$$\vec{L} = m\vec{r} \times \vec{v} = \vec{r} \times (m\vec{v}) = \vec{r} \times \vec{p} \tag{2}$$

The linear momentum is given by $\vec{p} = m\vec{v}$.

If we let θ be the angle between $\vec{r}$ and $\vec{r}$, then

$$L = mvr \sin\theta \tag{3}$$

If force is applied to an object, it can change the angular momentum.

$$\frac{d\vec{L}}{dt} = \vec{T} = \vec{r} \times \vec{F} \tag{4}$$

where $\vec{T}$ is the *torque*. If $\vec{T} = 0$, the angular momentum is constant, or conserved.

Let's see what the torque is for a central force.

$$\vec{T} = \vec{r} \times \vec{F} \tag{5}$$

$$\vec{T} = \vec{r} \times \left(\frac{GMm}{r^2}\right)\hat{r} \tag{6}$$

$$\vec{T} = \left(\frac{GMm}{r^2}\right)\vec{r} \times \hat{r} \tag{7}$$

Since $\vec{r} \times \hat{r} = 0$, there is no torque exerted by a central force, and the angular momentum is constant.

Let's see what the consequences are of conservation of angular momentum. **Look at Figure 3.3**. In a time interval dt the object of mass m moves, and sweeps out an area dA. We use the result from geometry that the area of a triangle is

$$dA = \frac{1}{2}bh \tag{8}$$

where $b = r$ and $h = v \sin\theta dt$, θ being the angle between $\vec{r}$ and $\vec{v}$.

So,

$$dA = \frac{1}{2}bh = \frac{1}{2}rv\sin\theta dt \tag{9}$$

And

$$\frac{dA}{dt} = \frac{1}{2}rv\sin\theta = \frac{L}{2m} \tag{10}$$

So $\frac{dA}{dt} = C$, a constant. This is Kepler's 2nd Law. A nice illustration of Kepler's 2nd Law is given in **Figure 2.18** of the book.

2.2 Kepler's 1st Law

Newton's Laws can be used to show that the equation for an orbit, given by $r(\theta)$, is described by Equation 3.34.

$$r(\theta) = \frac{L^2}{GMm^2(1 + e\cos\theta)} \tag{11}$$

The coordinates (r, θ) are *polar coordinates* centered on one of the focuses of the ellipse. The parameter e is very important, and is called the *eccentricity*. Its significance will be described shortly.

I think the derivation is too advanced for this course, but you can look at the discussion on pp66 and 67 for yourself.

This is the equation for geometric figures called *conic sections*, and include an ellipse, a parabola, or a hyperbola. Check out **Figure 3.5**.

Ellipses occur in the case of $0 \leq e < 1$. The geometry of an ellipse is shown in **Figure 3.6**. Be sure you known the following terms which are illustrated in this picture, or discussed in the text.

- major axis
- minor axis
- eccentricity
- focus
- true anomaly

2.2.1 Mean Size of an Orbit

While an ellipse is a precise description of an orbit, sometimes we want to know the mean (or average) distance of the object m from M. The astronomical unit is defined as the mean distance between the Sun and Earth. Let's work out what this is for an ellipse.

Look at **Figure 3.7**. Imagine putting dots along the orbit, and measuring the distance between the focus to each dot, and averaging the values. What is this number?

Let's imagine organizing these points into sets of two points, one at (x, y) and the other at $(-x, y)$, as shown in **Figure 3.8**. Let r be the distance from the right-hand focus to the first point, and $r^{'}$ the distance to the second point. The average distance of these two points is then $\frac{1}{2}(r + r^{'})$.

We could also choose the other, left-hand focus to make our measurements. In this case, the average distance of the two points from this second focus is $\frac{1}{2}(\rho + \rho^{'})$

Now, one of the definitions of an ellipse is that it is the set of points in a plane figure such that the sum of the distances from the point to the two focuses is the same number, say A. We then have that

$$r + \rho = A = r^{'} + \rho^{'} \tag{12}$$

If you look at **Figure 3.8** you can probably satisfy yourself that $r^{'} = \rho$, so we have

$$r + r^{'} = A \tag{13}$$

It only remains to determine the constant A. Since this constant applies to every point on the ellipse, we can choose any one we want to determine A. We can make life easy for ourselves by choosing the "perihelion" point on the x axis on the right side. You can easily show that $A = 2a$, so

$$\frac{1}{2}(r + r^{'}) = a \tag{14}$$

Notice that the mean distance doesn't depend on x or y. If it doesn't depend on (x,y) for these two points, it won't for an of the other points on the ellipse either, so we can conclude that the average distance for all points on the ellipse is a. The average distance of the object m from M is a.

3 Derivation of Kepler's 3rd Law

We want to derive the relationship between the semimajor axis and the period of the orbit. Follow the derivation on p72 and 73. Start with Kepler's 2nd Law,

$$\frac{dA}{dt} = \frac{L}{2m} \tag{15}$$

Since the RHS is constant, the total area swept out in an orbit is

$$A = \frac{L}{2m}P \tag{16}$$

We can use the standard expression for the total area of an ellipse, $A = \pi ab$, where b is the semimajor axis, to get the following.

$$\frac{\pi ab}{P} = \frac{L}{2m} \tag{17}$$

There is a relation between the semimajor axis, the semiminor axis, and the eccentricity, as follows,

$$b^2 = a^2(1 - e^2) \tag{18}$$

Now what do we do? Let's look at the equation for the ellipse we wrote down in Equation (11) above,

$$r(\theta) = \frac{L^2}{GMm^2(1 + e\cos\theta)} \tag{19}$$

This holds for every point on the ellipse, including *perihelion* and *aphelion*. At aphelion, $\theta = 0$ and $r = a(1-e)$, so we have

$$a(1-e) = \frac{L^2}{GMm^2(1+e)} \tag{20}$$

This means we have an expression for the angular momentum L in terms of the properties of the ellipse and the properties of the objects in the system.

$$a(1-e)GMm^2(1+e) = L^2 \tag{21}$$

$$L^2 = a(1-e^2)GMm^2 \tag{22}$$

$$\frac{L^2}{m^2} = a(1-e^2)GM \tag{23}$$

This is a result that we can substitute back into Equation (17). We first square both sides of Equation (17),

$$\frac{\pi^2 a^2 b^2}{P^2} = \frac{L^2}{4m^2} \tag{24}$$

$$\frac{\pi^2 a^2 a^2 (1-e^2)}{P^2} = \frac{a(1-e^2)GM}{4} \tag{25}$$

$$\frac{4\pi^2 a^3}{GM} = P^2 \tag{26}$$

Let's write Equation (26) out by itself.

$$\frac{4\pi^2 a^3}{GM} = P^2 \tag{27}$$

This is *exactly* Kepler's 3rd Law. The important result is that the period of the orbit depends only on the semimajor axis. The eccentricity doesn't enter into it. An orbit with an eccentricity of nearly zero and a semimajor axis of 1 au (like the Earth) has the same period as an object with an eccentricity of 0.99 and a semimajor axis of 1.00 au.

Dependence on the mass of the system In our equation (Equation 3.52) of the book, the relation between the period and the semimajor axis depends only on the mass M. We got this result because we assumed $M \gg m$, and assumed that m orbited the location of M. If we didn't make this approximation, we would find that M in our equation should be replaced with $(M+m)$. This would be the right expression to use in applying these results to binary stars.

4 The Two Body Problem

We have derived all three of Kepler's Laws from a theoretical physics system consisting of two objects interacting via gravity. This is remarkable, since the solar system consists of more than two objects. The reason that the two body description of planetary orbits works so well is that the force on each planet is dominated by the force between that planet and the Sun; all other forces are far weaker.

They are there, however, and result in *perturbations*, that cause properties of the orbits to slowly change over time. Properties of the Earth's orbit, for example, such as the eccentricity of its orbit, change on vary periodically with periods of tens of thousands of years.