General Astronomy (29:61)
Fall 2012
Lecture 9 Notes, September 14, 2012

1 Derivation of Kepler's 3rd Law

1.1 Derivation Using Kepler's 2nd Law

We want to derive the relationship between the semimajor axis and the period of the orbit. Follow the derivation on p72 and 73. Start with Kepler's 2nd Law,

$$\frac{dA}{dt} = \frac{L}{2m} \tag{1}$$

Since the RHS is constant, the total area swept out in an orbit is

$$A = \frac{L}{2m} P \tag{2}$$

We can use the standard expression for the total area of an ellipse, $A = \pi ab$, where b is the semimajor axis, to get the following.

$$\frac{\pi ab}{P} = \frac{L}{2m} \tag{3}$$

There is a relation between the semimajor axis, the semiminor axis, and the eccentricity, as follows,

$$b^2 = a^2(1 - e^2) \tag{4}$$

Now what do we do? Let's look at the equation for the ellipse we wrote down in Equation (11) above,

$$r(\theta) = \frac{L^2}{GMm^2(1 + e\cos\theta)} \tag{5}$$

This holds for every point on the ellipse, including *perihelion* and *aphelion*. At aphelion, $\theta = 0$ and $r = a(1 - e)$, so we have

$$a(1 - e) = \frac{L^2}{GMm^2(1 + e)} \tag{6}$$

This means we have an expression for the angular momentum L in terms of the properties of the ellipse and the properties of the objects in the system.

$$a(1 - e)GMm^2(1 + e) = L^2 \tag{7}$$

$$L^2 = a(1 - e^2)GMm^2 \tag{8}$$

$$\frac{L^2}{m^2} = a(1 - e^2)GM \tag{9}$$

This is a result that we can substitute back into Equation (17). We first square both sides of Equation (17),

$$\frac{\pi^2 a^2 b^2}{P^2} = \frac{L^2}{4m^2} \tag{10}$$

$$\frac{\pi^2 a^2 a^2 (1-e^2)}{P^2} = \frac{a(1-e^2)GM}{4} \tag{11}$$

$$\frac{4\pi^2 a^3}{GM} = P^2 \tag{12}$$

Let's write Equation (26) out by itself.

$$\frac{4\pi^2 a^3}{GM} = P^2 \tag{13}$$

This is *exactly* Kepler's 3rd Law.

2 Derivation for the Case of Circular Orbits

Let's do a different way of deriving Kepler's 3rd Law, that is only valid for the case of circular orbits, but turns out to give the correct result.

One justification for this approach is that a circle is a special case of an ellipse; one with zero eccentricity.

We let m be in a circular orbit around M. As it moves on its circle, the speed stays constant at v. Now let's apply the equation for the gravitational force,

$$\vec{F} = m\vec{a} = -\frac{GMm}{r^2}\hat{r} \text{ so} \tag{14}$$

$$\vec{a} = -\frac{GM}{r^2}\hat{r} \tag{15}$$

At first, you might think that an object moving on a circular path at a constant speed would not be accelerating. However, although the *magnitude* of the vector velocity is constant, its direction is constantly changing, so there is an acceleration. Look at the accompanying diagram to understand why there is an acceleration.

When an object moves through a curve, it feels an acceleration which points toward the center of the circle about which it is moving. This acceleration is called the *Centripetal Acceleration*.

The formula for the centripetal acceleration is

$$\vec{a}_c = -\omega^2 r\hat{r} \text{ with} \tag{16}$$

$$\omega = \frac{2\pi}{P} \text{ where } P \text{ is the orbital period} \tag{17}$$

$$\tag{18}$$

If we substitute this equation for the centripetal acceleration into the equation for the acceleration due to the gravitational acceleration, we have

$$\vec{a} = -\omega^2 r\hat{r} = -\left(\frac{2\pi}{P}\right)^2 r\hat{r} = -\frac{GM}{r^2}\hat{r} \tag{19}$$

$$\text{so....}\ \frac{4\pi^2}{P^2} = \frac{GM}{r^3} \text{ finally} \tag{20}$$

$$\left(\frac{4\pi^2}{GM}\right) r^3 = P^2 \tag{21}$$

Since $r = a$ for an ellipse, this is the same as Equation 3.52 in the book.

Although this expression was derived for the case of a circular orbit, *exactly* the same expression results for an ellipse with $e \neq 0$ (see Section 3.1.3 of the book). The important result is that the period of the orbit depends only on the semimajor axis. The eccentricity doesn't enter into it. An orbit with an eccentricity of nearly zero and a semimajor axis of 1 au (like the Earth) has the same period as an object with an eccentricity of 0.99 and a semimajor axis of 1.00 au.

Dependence on the mass of the system In our equation (Equation 3.52) of the book, the relation between the period and the semimajor axis depends only on the mass M. We got this result because we assumed $M \gg m$, and assumed that m orbited the location of M. If we didn't make this approximation, we would find that M in our equation should be replaced with $(M + m)$. This would be the right expression to use in applying these results to binary stars.

2.1 The Circular Orbit Equation

We can use the previous result to obtain a very handy formula that we can use throughout astronomy. It is correct for circular orbits, and can be used as an approximation for elliptical orbits. Let's start with our form of Kepler's 3rd Law.

$$\left(\frac{4\pi^2}{GM}\right) r^3 = P^2 \tag{22}$$

Let's change this to a formula relating the orbital speed to its size.

It is easy to see that

$$vP = 2\pi r \tag{23}$$

so

$$P = \frac{2\pi r}{v} \tag{24}$$

Substitute this expression for P into our above expression for Kepler's Law, and we have

$$\left(\frac{4\pi^2}{GM}\right) r^3 = \frac{4\pi^2 r^2}{v^2} \tag{25}$$

$$v^2 = \frac{GM}{r} \tag{26}$$

$$v = \sqrt{\frac{GM}{r}} \tag{27}$$

This is the circular orbit equation.

2.1.1 The orbital speed of the Earth around the Sun

Let's do an example. The semimajor axis of the Earth's orbit (mean distance of Sun and Earth) is 1 au = $1.496 \times 10^1 1$ m. The mass of the Sun is 1.989×10^{30} kg, so we have for the orbital speed of the Earth

$$v = \sqrt{\frac{6.673 \times 10^{-11}(1.989 \times 10^{30})}{1.496 \times 10^{11}}} \tag{28}$$

$$v = \sqrt{8.87 \times 10^8} \tag{29}$$

$$v = 2.97 \times 10^4 \text{ m/sec} \tag{30}$$

That's the right answer

3 Orbital Velocities for Elliptical Orbits

We can do one more exercise without a great deal of mathematical effort. We can calculate the orbital speed of an object at its fastest (at perihelion) and slowest (aphelion).

Let's go back to the equation for $r(\theta)$,

$$r(\theta) = \frac{L^2}{GMm^2(1 + e\cos\theta)} \tag{31}$$

This equation is true for every point on the elliptical orbit, including perihelion. Let's talk about perihelion, for which $\theta = 0$ and $r(0) = a(1-e)$, so

$$a(1-e) = \frac{L^2}{GMm^2(1+e)} \tag{32}$$

$$L^2 = GMm^2 a(1-e)(1+e) \tag{33}$$

Now let's think about the angular momentum $L = mvr\sin\theta$. At perihelion, $\theta = 90°$ (see diagram in online illustrations), so

$$L = mvr = mva(1-e) \tag{34}$$

so

$$L^2 = m^2v^2a^2(1-e)^2 = GMm^2a(1-e)(1+e) \text{ so} \tag{35}$$

$$v = \sqrt{\frac{GM(1+e)}{a(1-e)}} \tag{36}$$

Compare this with the circular orbit equation above.

The same approach works for aphelion, except $r = a(1+e)$. The result is

$$v = \sqrt{\frac{GM(1-e)}{a(1+e)}} \tag{37}$$

4 The Two Body Problem

We have derived all three of Kepler's Laws from a theoretical physics system consisting of two objects interacting via gravity. This is remarkable, since the solar system consists of more than two objects. The reason that the two body description of planetary orbits works so well is that the force on each planet is dominated by the force between that planet and the Sun; all other forces are far weaker.

They are there, however, and result in *perturbations*, that cause properties of the orbits to slowly change over time. Properties of the Earth's orbit, for example, such as the eccentricity of its orbit, change on vary periodically with periods of tens of thousands of years.