General Astronomy (29:61)
Fall 2012
Lecture 38 Notes, December 05, 2012

1 The Interior Structures of Jupiter and Saturn

Since Jupiter and Saturn are made of hydrogen and helium, this is a gas for the atmosphere that we see, and even deeper below the cloud layers where we can't see. Let's follow some mathematical arguments and see where this leads us.

The following is a variation of the derivation in Section 10.2.1 of the book.

When we discussed the atmosphere of the Earth, we discussed the property of *hydrostatic equilibrium*. As you go up higher in the atmosphere (larger value of altitude z), the change in the pressure is given by

$$\frac{dp}{dz} = -\rho g \tag{1}$$

Now we want to know how the pressure changes as we go outwards from the center of the planet to the surface and beyond, given by the radial coordinate r. If we are considering the change in pressure in spherical coordinates, can we just use the same formula as above and just swap in r for z?

It turns out in this case you can, although in general you can't be so quick in changing variables from Cartesian to spherical.

We then have an expression for the change in pressure as a function of distance r from the center of the planet,

$$\frac{dp}{dr} = -\rho g \tag{2}$$

It is easy enough to write down an expression for g. If the distribution of mass is spherically symmetric (a good approximation, even for rapidly rotating and flattened planets like Jupiter and Saturn), then

$$g = \frac{GM}{r^2} \tag{3}$$

Where $M = M(r)$ is the mass *interior* to a spherical surface of radius r. This equation tells us that $g = g(r)$, it is a function of r. M is given by the distribution of density (ρ) inside the planet.

Let's bring back our equation.

$$\frac{dp}{dr} = -\rho g \tag{4}$$

Now we really **do** seem to be stuck. To solve this equation for $p(r)$, we need to know how the density (ρ) varies with r. However, surely $\rho(r)$ depends on $p(r)$. How do we get out of this?

It turns out there are ways of approaching this problem.

However, for the moment, we will follow the example of the book and make a real gross approximation. It will give us some insight, and perhaps give us ballpark estimates for the value of the pressure in the interior of Jupiter and Saturn.

Let's assume that the density of Jupiter in the interior is uniform, and equal to the mean density $\bar{\rho}$. The total mass is given by

$$\frac{4}{3}\pi R^3 \bar{\rho} = M \tag{5}$$

$$\bar{\rho} = \frac{3M}{4\pi R^3} \tag{6}$$

In the interior of the planet,

$$g(r) = \frac{GM(r)}{r^2} \tag{7}$$

$$M(r) = \frac{4}{3}\pi r^3 \bar{\rho} \text{ so} \tag{8}$$

$$g(r) = \frac{4\pi G\bar{\rho}r^3}{3r^2} \text{ and} \tag{9}$$

$$\frac{dp}{dr} = -\frac{4\pi}{3}G\bar{\rho}^2 r \tag{10}$$

This equation tells us that in this approximation, the pressure drops rapidly as we approach the edge of the planet. $\longrightarrow$ drawing on blackboard.

We can solve this equation for the pressure as a function of r. This requires that we use integrals from calculus

$$\frac{dp}{dr} = -\frac{4\pi}{3}G\bar{\rho}^2 r \tag{11}$$

$$dp = -\frac{4\pi}{3}G\bar{\rho}^2 r dr \tag{12}$$

Let's integrate this from $r = 0$, where $p = p_c$, to $r = R$, where $p = 0$.

$$dp = -\frac{4\pi}{3}G\bar{\rho}^2 r dr \tag{13}$$

$$\int_{p_c}^{0} dp = -\frac{4\pi}{3}G\bar{\rho}^2 \int_0^R r dr \tag{14}$$

$$0 - p_c = -\frac{4\pi}{3} G\bar{\rho}^2 \left(\frac{1}{2} R^2\right) \tag{15}$$

$$p_c = \frac{2\pi}{3} G\bar{\rho}^2 R^2 \tag{16}$$

So this is our answer, with all its inadequacies, but still capable of providing us with some insight,

$$p_c = \frac{2\pi}{3} G\bar{\rho}^2 R^2 \tag{17}$$

Let's plug in some numbers and see what we get for Jupiter. We know that $R = 10.97 R_\oplus = 10.97(6.37 \times 10^6)$, and $\bar{\rho} = 1.33$ g-cm $^{-3} = 1.33^3$ SI.

So we have

$$p_c = \frac{2\pi}{3} G\bar{\rho}^2 R^2 \tag{18}$$

$$p_c = \frac{2\pi}{3} 6.673 \times 10^{-11} (1.33 \times 10^3)^2 (10.97(6.37 \times 10^6))^2 \tag{19}$$

$$p_c = 1.21 \times 10^{12} \text{ N/m}^2 \tag{20}$$

This is about 10^7 times the pressure of the Earth's atmosphere. At these extreme pressures, matter can take on some strange properties. As you approach this pressure, hydrogen first becomes *liquid*, and at pressures of order a million atmospheres, becomes *liquid metallic hydrogen*. Hydrogen becomes a liquid metal like mercury.

$\longrightarrow$ Look at Figure 10.11 of your book, it shows the *phase diagram* of hydrogen

We can use this basic knowledge of physics to conclude that the interior of Jupiter and Saturn consist of this bizarre form of matter. Think of this when you look up in the sky at Jupiter!