Numerical analysis with wavelets

W. N. Polyzou

Other contributors: Gerald Payne, Brian Kessler, Fatih Bulut, Tracie Michlin

The University of Iowa

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Outline

- Background and motivation.
- Fractals self-similarity and the renormalization group.
- Numerical analysis using the renormalization group.
- Applications.

Standard numerical methods

- **Designed to work with functions that with enough magnification locally look like straight lines.**

- **Global basis functions $\{\phi_n(x)\}$ are often associated with a scale. They may have to be rescaled $\{\sqrt{a}\phi_n(ax)\}$ to efficiently represent structures on a given scale.**

$$f(x) = \sum_n f_n \phi_n(x) \qquad f_n = \int f(x)\phi_n^*(x)dx$$

- **Global basis functions (orthoginal polynomials, Laguerre functions, Hermite functions) can have difficulties dealing with local structures, or structures on multiple scales.**

Motivation for considering wavelets

- Used for data compression in digital photography (JPEG).
- Efficient at treating images with many different scales and structures.
- A digital photograph is just a matrix of numbers. Could this same data compression method be used to efficiently solve problems in linear algebra?
- Wavelet bases result in sparse matrices. Faster algorithms can be used and they require less storage.

Challenges

- **Wavelets are fractal valued functions**
- **How do you evaluate fractal functions $\{\phi_n(x)\}$?**

$$f(x) = \sum f_n \phi_n(x)$$

- **How do you calculate integrals involving fractal functions?**

$$f_n = \int \phi_n^*(x) f(x) dx$$

- **How do you calculate derivatives of fractal functions?**

$$f''(x) = \sum f_n \phi_n''(x)$$

- **Why would you want to use them?**

What do we mean by a fractal valued function

- **Looks like a copy of itself on smaller scales.**

- **How do we change scales mathematically?**

$$Df(x) = \sqrt{2}f(2x)$$

- **Shrinks the support of the function by a factor of 2, preserving the Hilbert space norm of the function.**

- D **is called a scaling of dilatation operator**

Renormalization group equation solutions $s(x)$ are fractal!

$$\underbrace{s(x) = D\ \underbrace{(\sum_{l=0}^{2K-1} h_l T^l s(x))}_{\text{weighted average}}}_{\text{rescale}}\ .$$

T: unit translation **D scale transformation**

$$Ts(x) = s(x-1) \qquad Ds(x) = \sqrt{2}s(2x).$$

The renormalization group equation is homogeneous
$s(x)$ a solution implies $cs(x)$ is a solution
the scale c is fixed by

$$\int dxs(x) = 1$$

Properties of $s(x)$

$$\tilde{s}(k) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} s(x) dx \qquad \tilde{h}(k) := \sum_l \frac{h_l}{\sqrt{2}} e^{-ikl}.$$

$$\tilde{s}(k) = \tilde{s}\left(\frac{k}{2}\right) \tilde{h}\left(\frac{k}{2}\right)$$

$$\tilde{s}(k) = \lim_{n\to\infty} \tilde{s}(\frac{k}{2^n}) \prod_{l=1}^{n} \tilde{h}(\frac{k}{2^l}) = \tilde{s}(0) \prod_{l=1}^{\infty} \tilde{h}(\frac{k}{2^l}).$$

$$k = 0 \qquad \to \qquad 1 = \prod_{l=1}^{\infty} \tilde{h}(0) = \tilde{h}(0) = \sum_l \frac{h_l}{\sqrt{2}}$$

$\sum_l h_l = \sqrt{2}$: Necessary for the renormalization group equation to have a solution

Support of the solution $s(x)$ to the RG equation

$$s(x) = \frac{\tilde{s}(0)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \prod_{m=1}^{\infty} \tilde{h}(\frac{k}{2^m}) =$$

$$\sqrt{2\pi}\tilde{s}(0) \prod_{m=1}^{\infty} (\sum_{l=0}^{N-1} \frac{h_l}{\sqrt{2}} \delta(x - \frac{l}{2^m}))$$

which vanishes for $x \notin [0, N-1]$.

Support of $s(x) \in [0, N-1]$ where N is the number of h_l's

$N = 2K$

All fractals are not created equal

Additional properties of $s(x)$ depend on the choice of h_l

$$\sum_{l=1}^{2K-1} h_l = \sqrt{2}$$

Additional conditions:

$$\int s(x-n)s(x-m)dx = \delta_{mn}$$

$$\int x^m \sum_{l=0}^{2K-1} (-)^l h_{2K-l-1} s(x-l) = 0 \qquad m = 0, 1, \cdots K-1$$

Conditions define Daubechies K scaling functions:

Meaning of conditions:

- $s_n(x) := T^n s(x) = s(x - n)$ **are orthonormal.**

- $x^m = \sum_n c_n s_n(x)$ **pointwise for** $m \leq K$**.**

- **Equations determine** h_l **up to reflection:**
  $h_l \to h'_l = \_2k - 1 - l$**. See table:**

Weight coefficients for different K values

h_l	K=1	K=2	K=3
h_0	$1/\sqrt{2}$	$(1+\sqrt{3})/4\sqrt{2}$	$(1+\sqrt{10}+\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_1	$1/\sqrt{2}$	$(3+\sqrt{3})/4\sqrt{2}$	$(5+\sqrt{10}+3\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_2	0	$(3-\sqrt{3})/4\sqrt{2}$	$(10-2\sqrt{10}+2\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_3	0	$(1-\sqrt{3})/4\sqrt{2}$	$(10-2\sqrt{10}-2\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_4	0	0	$(5+\sqrt{10}-3\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_5	0	0	$(1+\sqrt{10}-\sqrt{5+2\sqrt{10}})/16\sqrt{2}$

Multi-resolution decomposition of $L^2(\mathbb{R})$

Rescale and translate fixed point, $s(x)$

$$s_n^k(x) := D^k T^n s(x) = 2^{k/2} s\left(2^k(x - 2^{-k}n)\right).$$

$\mathcal{S}_k :=$ **resolution** 2^{-k} **subspace of** $L^2(\mathbb{R})$**,** $\{s_n^k(x)\}$ **basis**

$$\mathcal{S}_k := \{f(x) | f(x) = \sum_{n=-\infty}^{\infty} c_n s_n^k(x), \quad \sum_{n=-\infty}^{\infty} |c_n|^2 < \infty\}.$$

$$\mathcal{S}_k := D^k \mathcal{S}_0$$

$$\mathcal{S}_k \subset \mathcal{S}_{k+n} \qquad n \geq 0$$

$$\mathcal{S}_{k+1} = \mathcal{S}_k \oplus \mathcal{W}_k \qquad \mathcal{W}_k \neq \{\emptyset\}.$$

Multi-resolution decomposition of $L^2(\mathbb{R})$

$$\mathcal{S}_{k+1} = \mathcal{S}_k \oplus \mathcal{W}_k$$

$$L^2(\mathbb{R}) = \mathcal{S}_k \oplus \mathcal{W}_k \oplus \mathcal{W}_{k+1} \oplus \mathcal{W}_{k+2} \oplus \mathcal{W}_{k+3} \oplus \cdots =$$

$$\cdots \oplus \mathcal{W}_{k-2} \oplus \mathcal{W}_{k-1} \oplus \mathcal{W}_k \oplus \mathcal{W}_{k+1} \oplus \mathcal{W}_{k+2} \oplus \cdots$$

Wavelets ($\{w_n^k(x)\}$ orthonormal basis for $\mathcal{W}_k$)

$$w(x) := \sum_{l=0}^{2K-1} g_l T^l s(x) \qquad g_l = (-)^l h_{2K-1-l}$$

$$w_n^k(x) := D^k T^n w(x) = 2^{k/2} w\left(2^k(x - 2^{-k} n)\right).$$

Comments

Orthonormal basis for $L^2(\mathbb{R})$:

$$\{s_n^k(x)\}_{n=-\infty}^{\infty}\} \cup \{w_n^l(x)\}_{n=-\infty, l\geq k}^{\infty}$$

The condition

$$\int x^m \sum_{l=0}^{2K-1} (-)^l h_{2K-l-1} s(x-l) = 0 \qquad m = 0, 1, \cdots K$$

that determines the h_l is equivalent to the requirement

$$\int x^m w_n^k(x) dx = 0 \qquad \forall k, n \qquad \text{and} \qquad m = 0, 1, \cdots, K-1$$

Comments

Completeness implies

$$f(x) = \sum_n c_n s_n^k(x) + \sum_{n,l \geq k} d_{nl} w_n^l(x)$$

For $f(x) = x^m$, $m < K$ all the $d_{nl} = 0$ which means

$$x^m = \sum_n c_n s_n^k(x) \qquad m < K$$

where for any x only a finite number of the $s_n^k(x)$ are non-zero.

This requires that both sides of the equation agree at every x

- **This explains why JPEG works. All of coefficients d_n^k associated with structures that are smooth on scale 2^{-k} vanish, or are very small, resulting in an efficient representation of the data.**

- **The transformation relating $\mathcal{S}_{k+n}$ and $\mathcal{S}_k \oplus \mathcal{W}_k \oplus \cdots \oplus \mathcal{W}_{k+n}$ is a real orthogonal transformations, called the wavelet transform, that can be performed very efficiently.**

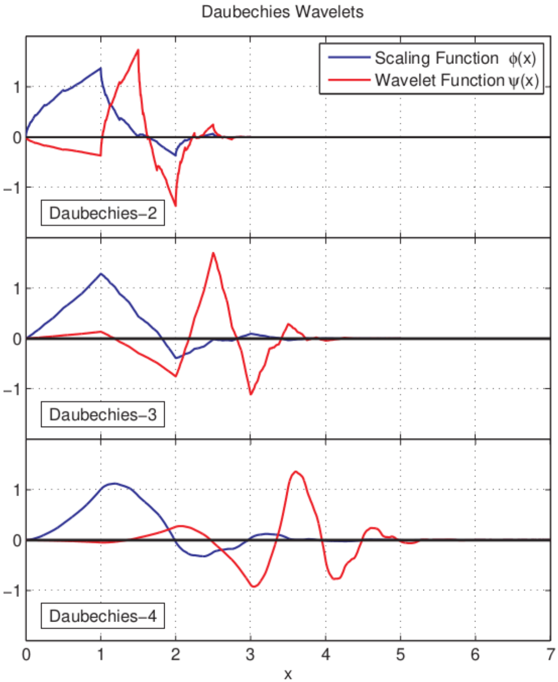
- **The wavelet transformation can be used to recover an approximation to the original function.**

Wierd stuff!

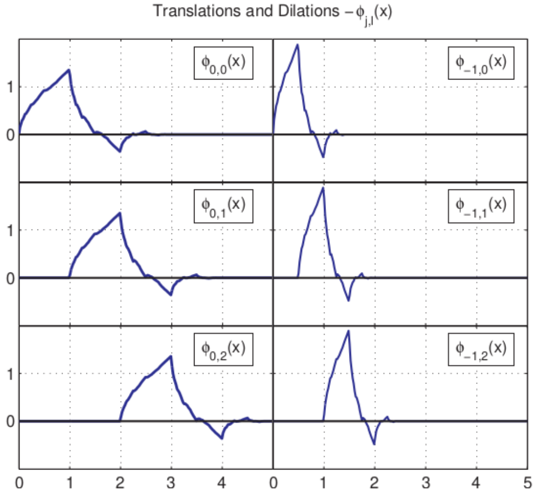
$$1 = \int s(x)dx = \int s^2(x)dx$$

- **Locally finite sums of fractal functions are locally infinitely differentiable. This means that a FINITE number of these functions with complex fractal boundaries fit together like a jigsaw puzzle!**

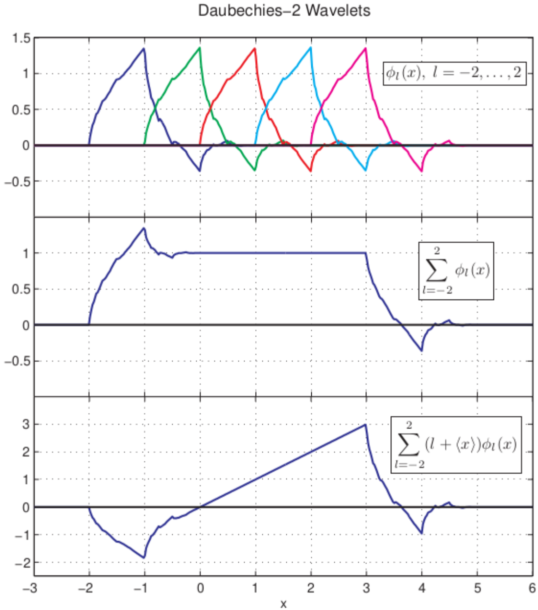
- **Under any magnification the functions do not look like straight lines, but they are differentiable!**



Daubechies Wavelets
Scaling Function φ(x)
Wavelet Function ψ(x)
Daubechies−2
Daubechies−3
Daubechies−4
1
0
−1
0
1
2
3
4
5
6
7
x

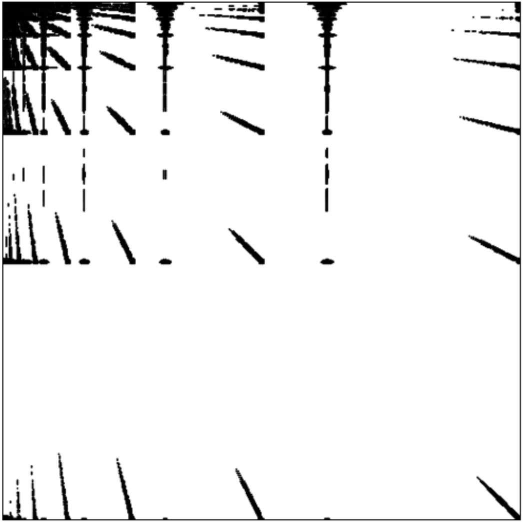


Translations and Dilations – $\phi_{j,l}(x)$
$\phi_{0,0}(x)$
$\phi_{-1,0}(x)$
$\phi_{0,1}(x)$
$\phi_{-1,1}(x)$
$\phi_{0,2}(x)$
$\phi_{-1,2}(x)$
1
0
1
0
1
0
0 1 2 3 4 0 1 2 3 4 5



Daubechies-2 Wavelets
$\phi_l(x),\ l=-2,\ldots,2$
$\sum_{l=-2}^{2}\phi_l(x)$
$\sum_{l=-2}^{2}(l+\langle x\rangle)\phi_l(x)$
x

Kernel of a scattering integral equation in the wavelet basis



Computation of fractal basis functions?

Approximation by iteration.
Pick any function $s(1;x)$ satisfying $\int s(1;x)dx = 1$.

$$s(n;x) = \sum_l h_l D T^l s(n-1;x)$$

$$s(x) = \lim_{n\to\infty} s(n;x)$$

Exact calculation at dyadic rationals,
use the renormalization group equation at dyadic rational points:

$$s(n) = \sqrt{2}\sum_l h_l s(2n-l) \qquad \sum_n s(n) = 1 \qquad n = 1, 2, 2K-2$$

$$s(r/2) = \sqrt{2}\sum_l h_l s(r-l) \qquad r = \frac{m}{2^k}$$

Calculations of integrals use the renormalization group equation and the scale fixing condition.

All moments of $s_n^k(x)$ and $w_n^l(x)$ can be computed exactly.

$$< x^m >_{s_n^k} := \int x^m s_n^k(x) dx$$

$$< x^m >_{w_n^k} := \int x^m w_n^k(x) dx$$

Calculating moments:

$$< x^m >_s = \int s(x) x^m dx \qquad < x^m >_w = \int w(x) x^m dx.$$

$$< x^0 >_s = (x^0, s) = \int dx s(x) = 1$$

Using the renormalization group equation:

$$< x^m >_s = (x^m, s) = (D^{-1} x^m, D^{-1} s)$$

$$= \frac{1}{\sqrt{2}} \frac{1}{2^m} \sum_l h_l (x^m, T^l s)$$

$$= \frac{1}{\sqrt{2}} \frac{1}{2^m} \sum_l h_l ((x+l)^m, s)$$

$$= \frac{1}{\sqrt{2}} \frac{1}{2^m} \sum_l h_l \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} l^{m-k} < x^k >_s .$$

Using $\sum_l h_l = \sqrt{2}$, and moving the $k = m$ term to the left side of the above equation gives the recursion relation:

$$< x^m >_s = \frac{1}{2^m - 1} \frac{1}{\sqrt{2}} \sum_{k=0}^{m-1} \frac{m!}{k!(m-k)!} \left(\sum_{l=1}^{2K-1} h_l l^{m-k} \right) < x^k >_s \; .$$

$$< x >_s := \int x s(x) dx = \frac{1}{\sqrt{2}} \sum_{l=0}^{2K-1} l h_l .$$

With this definition we have:

$$\int (a + bx) s(x) dx = a + b < x >_s \; .$$

Boundary conditions:

$$< x^m >_{s:0} = \int_0^\infty x^m s(x) dx$$

The renormalization group equation relates these endpoint moments to ordinary moments.

The computation reduces to linear algebra.

Integrating functions with small support

One-point quadrature

$$\sum_m ms(x-m) = x - \langle x \rangle$$

which gives

$$0 = \sum_m m \int xs(x)s(x-m)dx = \sum_m \int s(x)x(x - \langle x \rangle_s)dx$$

$$= \langle x \rangle_s^2 - \langle x \rangle_1^2 \; .$$

This means that $\langle x^2 \rangle_s = \langle x \rangle_s^2$ **or**

$$\int s(x)p(x)dx = p(\langle x \rangle_s) \qquad p(x) = a + bx + cx^2.$$

General methods:

Increasing resolution 2^{-k}:

$$s_n^k(x) = \sum_{l=0}^{2K-1} h_l s_{2n+l}^{k+1}(x)$$

Replacing wavelets by scaling functions:

$$w_n^k(x) = \sum_{l=0}^{2K-1} g_l s_{2n+l}^{k+1}(x)$$

Changing scale: $2^{-k} \to 0$

$$\int s_{n_1}^k(x) \cdots s_{n_m}^k(x) dx = 2^{\frac{km}{2} - k} \int s_{n_1}^0(x) \cdots s_{n_m}^0(x) dx$$

Calculating the $k = 0$ integrals

Use the renormalization group equation

$$\Gamma_{n_1,\cdots,n_k} := \int s^0_{n_1}(x)\cdots s^0_{n_m}(x)dx$$

Homogeneous equation (RG equation):

$$\Gamma_{n_1,\cdots n_m} = \sum 2^{m/2-1} h_{l_1}\cdots h_{l_m}\Gamma_{2n_1+l_1,\cdots,2n_m+l_m}$$

Inhomogeneous equation: use $\sum s_n(x) = 1$:

$$\sum_{n_1} \Gamma_{n_1,\cdots n_m} = \Gamma_{n_2,\cdots n_m}$$

Solve the finite linear system

Products of scaling functions and polynomials

Homogeneous equations (RG equation)

$$I^m_{n_1\cdots n_k} := \int x^m s_{n_1}(x)\cdots s_{n_k}(x)dx =$$

$$2^{-\frac{2m+k}{2}} \sum h_{l_1}\cdots h_{l_k} \int x^m s_{2n_1+l_1}(x)\cdots s_{2n_k+l_k}(x)dx =$$

$$2^{-m-k/2} \sum h_{l_1}\cdots h_{l_k} I^m_{2n_1+l_1,\cdots 2n_k+l_k}$$

Inhomogeneous equations

$$\sum_{n_1} I^m_{n_1\cdots n_k} = I^m_{n_2\cdots n_k}$$

Integrals with derivatives

$$\frac{ds(x)}{dx} = 2\sum h_l D T^l \frac{ds(x)}{dx}$$

Replaces renormalization group equation

$$x = < x >_s + \sum_n n s_n(x)$$

Normalization condition

$$1 = \sum_n n \frac{ds_n(x)}{dx}$$

A necessary condition for the solution of the scaling equation to have k derivatives can be obtained by differentiating the scaling equation k times, which gives

$$s^{(k)}(x) = \sqrt{2}2^k \sum_l h_l s^{(k)}(2x - l)$$

Letting $x = m$ and $n = 2m - l$ gives the eigenvalue equation

$$s^{(k)}(m) = \sqrt{2}2^k \sum_n h_{2m-n} s^{(k)}(n)$$

$$\sum_n H_{mn} s^{(k)}(n) = 2^{-k-\frac{1}{2}} s^{(k)}(m)$$

The matrix is $(2K - 2) \times (2K - 2)$ which limits the number of eigenvalues (again - calculus replaced by linear algebra).

Singular integrals

$$S_{n+} := \int \frac{s_n(x)dx}{x + i0^+}$$

Renormalization group equation

$$S_{n+} := \sqrt{2} \sum_l h_l S_{2n-l+}$$

Treatment of singularity

$$-i\pi = \sum_n \int_{-a}^{a} \frac{dx s_n(x)}{x + i0^+} = \sum_n S_{n:a}$$

$$S_{n:a+} = \int_{-a}^{a} \frac{s_n(x)dx}{x + i0^+} = \int_{-a-n}^{a-n} \frac{s(x)dx}{1 + x/n} =$$

$$\frac{1}{n} \sum_{k=0}^{\infty} (\frac{-1}{n})^k \int_{-a-n}^{a-n} x^k s(x)dx$$

RG equation couples integrals over $x =$ to integrals with support far from $x = 0$. The integrals far from the singularity can be approximated in terms of moments. Rigorous error bounds can be computed. The evaluation of the integrals reduce to algebra.

$$S_{n+} = \int \frac{s_n(x)dx}{x + i0^+} \qquad n = -1, -2, -3, -4$$

Table 2: Singular integrals $(K = 3)$

S_{-1+}	-0.1717835441734- i4.041140804162
S_{-2+}	-1.7516314066967+ i1.212142562305
S_{-3+}	-0.3025942645356- i0.299291822651
S_{-4+}	-0.3076858066180- i0.013302589081

Integrals with natural logs

$$L(n) := \int_0^\infty s_n(x)\ln(x)dx$$

The renormalization group equations gives

$$L(n) = \frac{1}{\sqrt{2}}\left(\sum_l h_l L(2n+l) - \ln(2)\right).$$

$L(n)$ **for large n can be expressed in terms of moments**

$$L(n) = \int s_n(x)\ln(x)dx = \int s(y)\ln(n(1+y/n))dy$$

$$= \ln(n) - \sum_{m=1}^{\infty}\frac{(-1)^m}{m}\frac{< x^m >_s}{n^m}.$$

Table 2: Log integrals

$K=2$		
$n=-2$	$\int s_n(x)\ln\lvert x\rvert dx$	0.456927033732831
$n=-1$	$\int s_n(x)\ln\lvert x\rvert dx$	-1.64215549088219
$K=3$		
$n=-4$	$\int s_n(x)\ln\lvert x\rvert dx$	1.15737952417967
$n=-3$	$\int s_n(x)\ln\lvert x\rvert dx$	0.750468355278047
$n=-2$	$\int s_n(x)\ln\lvert x\rvert dx$	0.315624303943019
$n=-1$	$\int s_n(x)\ln\lvert x\rvert dx$	-1.83646456399118

Autocorrelation function

$$A(x) := \int s(x-y)s(y)dy$$

Renormalization group equation for $A(x)$

$$A(x) = \sum_{m,n} h_m h_n A(2x-m-n)$$

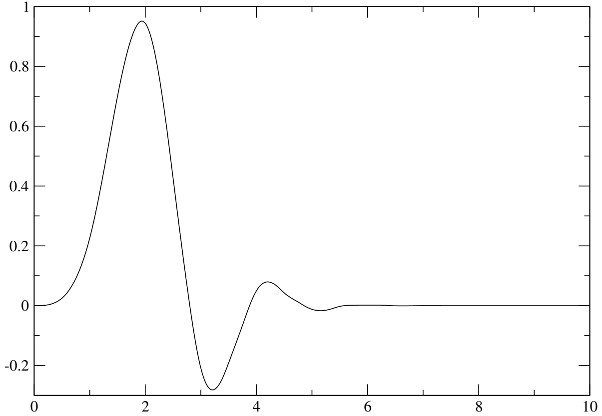
$$a_l = \frac{1}{\sqrt{2}} \sum_n h_{l-n} h_n$$

$$A(x) = \sum_{l=0}^{4K-2} a_l D T^l A(x)$$

Scale fixing for A(x)

$$\int A(x) = 1$$

Autocorrelation function A(x)



Integrals with moving singularities

$$J_{k-m-n} := \int \frac{s_m(x)s_n(y)dxdy}{k-x-y+i0^+} = \int \frac{A(x)dx}{k-m-n-x+i0^+}$$

Renormalization group equation for $A(x)$

$$J_n = \sqrt{2}\sum_l a_l J_{2n-l}$$

$$1 = \sum_n A(x+n)$$

$$i\pi = \sum_n \int_{-a}^{a} \frac{A(x+n)dx}{-x+i0^+} = \sum_n J_{n:a}$$

Applications

- **Solving scattering integral equations.**

- **Eliminating short distance degrees of freedom in quantum field theory.**

- **Exact discrete representations of quantum field theory.**

Conclusions

- Daubechies wavelets are a useful basis for problems involving multiple scales.
- Standard numerical methods do not work very well when applied to fractal functions.
- In most cases the standard numerical methods can be replaced by new methods based on the renormalization group equation.
- The calculations of derivatives and integrals are replaced by linear algebra.
- Basis provides local control over resolution while remaining efficient.

XXX

Example: $D_{\mathbf{mn}}$

0 unless the support of s_m and s_n overlap

$$D_{mn} = D_{m-n,0} = \int \frac{ds_m(x)}{dx}\frac{ds_n(x)}{dx}dx$$

non-zero solutions have exact rational values

$$D_{40} = D_{-40} = -3/560$$

$$D_{30} = D_{-30} = -4/35$$

$$D_{20} = D_{-20} = 92/105$$

$$D_{10} = D_{-10} = -356/105$$

$$D_{00} = 295/56.$$

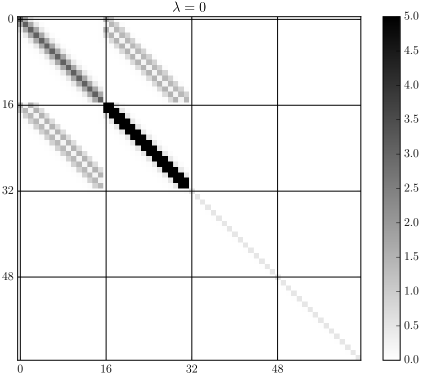
$$U(\lambda) = e^{K(\lambda)}, \qquad K(\lambda) = [G(\lambda), H(\lambda)]$$

where $G(\lambda)$ is the part of $H(\lambda)$ with the operators that couple different scales turned off. With this choice $G(\lambda) = G^\dagger(\lambda)$ so $K(\lambda)$ is anti-Hermetian.

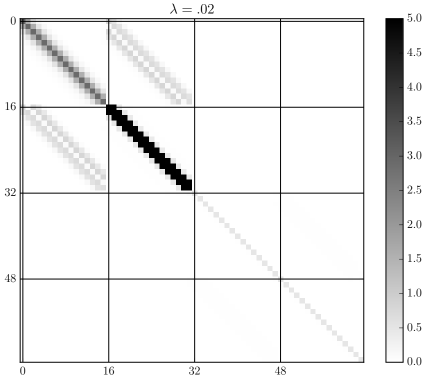
It follows that

$$\frac{dH(\lambda)}{d\lambda} = [K(\lambda), H(\lambda)] = [H(\lambda), [H(\lambda), G(\lambda)]].$$

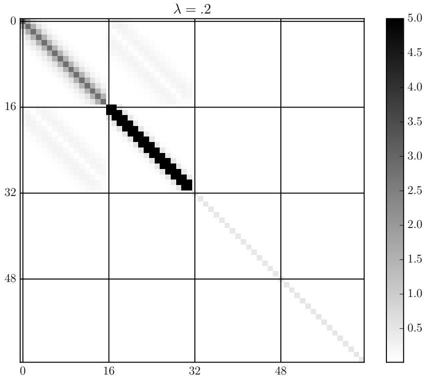
$H(\lambda)$ **is block diagonal when** $[H(\lambda), G(\lambda)] = 0$**.**

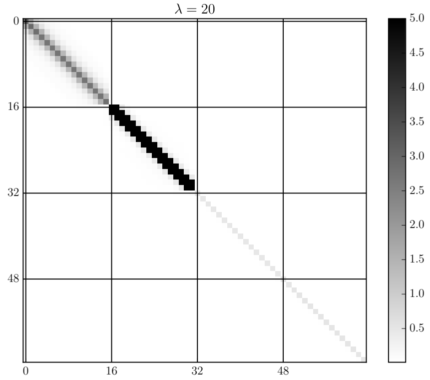


$\lambda = 0$
0
16
32
48
0
16
32
48
5.0
4.5
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0



$\lambda = .02$
0
16
32
48
0
16
32
48
5.0
4.5
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0


$\lambda = .2$
0
16
32
48
0
16
32
48
5.0
4.5
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5



$\lambda = 20$
0
16
32
48
0
16
32
48
5.0
4.5
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5

Approximation Space

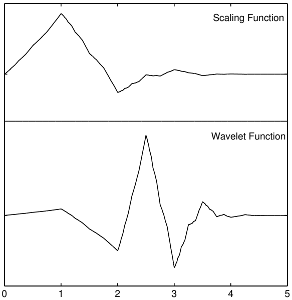
- For large m and smooth $f(x)$

$$f(x) \approx \sum_n f_n \phi_{-mn}(x)$$

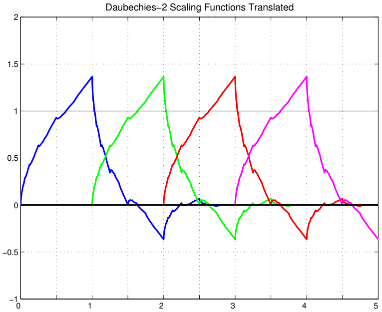
$$f_n = \int \phi_{-mn}(x) f(x) dx \approx 2^{-m/2} f(\frac{n}{2^m})$$

- Expansion coefficients proportional to function values on support of ϕ_{-mn}.

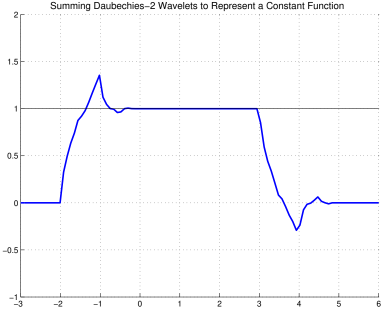
$K = 3$ Wavelet and Scaling Function



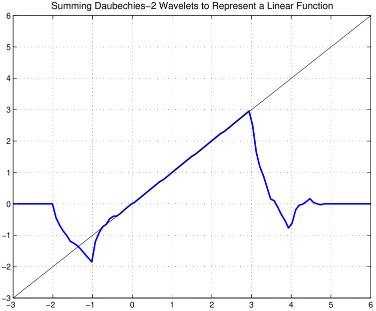
Unit Translates



Constant Function



Linear Function



Local Polynomials

$$L^2(R) = \cdots \oplus \mathcal{W}_{m-2} \oplus \mathcal{W}_{m-1} \cdots \oplus \mathcal{W}_m \oplus \mathcal{V}_m$$

$$\int \psi_{mn} x^l = 0 \qquad \forall m, n; \quad l = 0, 1, \cdots, K-1$$

$$\Downarrow$$

$$\therefore \qquad x^l = \sum_n c_n \phi_{mn}(x) \qquad \text{pointwise}$$

Computation of $\phi(x)$ and $\psi(x)$

- $\phi(x)$ at integer points can be obtained by solving:

$$\phi(n) = \sum_{l=0}^{2K-1} \sqrt{2} h_l \phi(2n - l) \qquad \sum_n \phi(n) = 1$$

- The support of $\phi(x) \in [0, 2K - 1]$.

$$\Downarrow$$

$$\phi(n) = 0 \qquad n \leq 0 \quad \text{or} \quad n \geq 2K - 1$$

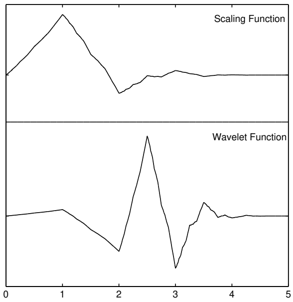
Computation of $\phi(x)$ and $\psi(x)$

- The scaling equation generates ψ and ϕ recursively at all dyadic rationals:

$$\phi(\frac{n}{2^k}) = \sum_{l=0}^{2K-1} \sqrt{2} h_l \phi(\frac{n}{2^{k-1}} - l)$$

$$\psi(\frac{n}{2^k}) = \sum_{l=0}^{2K-1} \sqrt{2} g_l \phi(\frac{n}{2^{k-1}} - l)$$

$K = 3$ Wavelet and Scaling Function



Approximation Spaces

- There are two approximation spaces related by a fast orthogonal transformation ($o(N)$).

$$\mathcal{V}_m \Leftrightarrow \mathcal{W}_{m+1} \oplus \mathcal{W}_{m+2} \oplus \cdots \oplus \mathcal{W}_{m+k} \oplus \mathcal{V}_{m+k}$$

with orthonormal bases

$$\{\phi_{mn}(x)\}_{n=-\infty}^{\infty}$$

$$\Updownarrow$$

$$\{\phi_{m+k\,n}(x)\}_{n=-\infty}^{\infty} \cup \{\psi_{ln}(x)\}_{n=-\infty,l=m+1}^{\infty,m+k}$$

Interesting Properties

- Wavelets are fractals.
- Basis functions are generated from a single "Mother" function by translations and dyadic scale changes.
- "Mother" function constructed from solution ("Father" function = scaling function) of a linear renormalization-group equation.

Useful Numerical Properties

- Basis functions have compact support.
- Basis functions are orthonormal.
- Basis functions never have to be computed.
- The wavelet transform automatically eliminates unimportant basis functions.
- Basis functions can locally pointwise represent polynomials.
- Wavelets lead to efficient treatment of scattering singularities.
- There is an efficient one-point quadrature rule.

Why are Wavelets Interesting ?

- Efficient representation of information.
- Used in the FBI's fingerprint archive.
- Used in the JPEG2000 image compression algorithm.
- Fast reconstruction of information.
- Natural basis for functions with smooth structures on multiple scales.

Structure of equation:

$$f(x) = g(x) + \int \frac{K(x,y)}{y} f(y) dy$$

$$f = \sum f_n \phi_n(x) \qquad x_n = \langle x^1 \rangle_{\phi_n}$$

$$f_m = g(x_m) + \sum_n \left(\frac{K(x_m, y_n) - K(x_m, 0)}{y_n} + K(x_m, 0) I_n \right) f_n$$

$$f(x) = g(x) + \sum_n \left(\frac{K(x, y_n) - K(x, 0)}{y_n} + K(x, 0) I_n \right) f_n$$

Transformed Kernel